A note on generalized Möbius μ-functions

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In [1] the concept of a conjugate pair of sets of positive integers is introduced. Briefly, if Z denotes the set of positive integers and P and Q denote non-empty subsets of Z such that: if $n_1 \in Z$, $n_2 \in Z$, $(n_1, n_2) = 1$, then

$$n = n_1 n_2 \in P \text{ (resp. } Q) \Longleftrightarrow n_1 \in P, n_2 \in P (\text{resp. } Q) \ , \tag{1}$$

and, if in addition, for each integer $n \in Z$ there is a unique factorization of the form

$$n = ab, \ a \in P, \ b \in Q \ , \tag{2}$$

we say that each of the sets P and Q is a *direct factor set* of Z, and that (P, Q) is a *conjugate pair*. It is clear that $P \cap Q = \{1\}$. Among the generalized functions studied in [1], we find

$$\mu_P(n) = \sum_{\substack{d|n \\ d \in P}} \mu\left(\frac{n}{d}\right)$$

a generalization of Möbius μ-function. The following results are also proved in [1]:

(i) μ_p is a multiplicative function.

(ii)
$$\sum_{\substack{d|n \\ d \in Q}} \mu_P\left(\frac{n}{d}\right) = \rho(n) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1 \end{cases}$$

Here we shall show that μ_P is the unique arithmetical function satisfying (ii) above. Let μ^* be such that

$$\sum_{\substack{d|n \\ d \in Q}} \mu^*\left(\frac{n}{d}\right) = \rho(n) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1 \end{cases} \tag{4}$$

If μ^* is multiplicative, it suffices to prove that $\mu_P(p^k) = \mu^*(p^k)$ for every prime p and every integer > 0. So let μ^* be a multiplicative function; it follows from (4) and (ii) that

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(*) Publicado en la *Revista Colombiana de Matemáticas*, **2** (1968), 6–11. [MR: 38 5699] .

$\mu_P(1) = \mu^*(1)$, thus $\mu_P(p) = \mu^*(p)$ for every prime p. We will now show by induction on k that $\mu_P(p^k) = \mu^*(p^k)$. Suppose this relation holds for $u > k \geq 0$. From (ii) we obtain

$$\mu_P(p^u) = -\sum_{\substack{1 \leq i \leq u \\ p^i \in Q}} \mu_P\left(\frac{p^u}{p^i}\right) \ , \tag{5}$$

because $1 = p^0 \in Q$. On the other hand, from (4) we obtain

$$\mu^*(p^u) = -\sum_{\substack{1 \leq i \leq u \\ p^i \in Q}} \mu^*\left(\frac{p^u}{p^i}\right) \ , \tag{6}$$

because $1 = p^0 \in Q$. But by the induction hypothesis $\mu_P(p^{u-i}) = \mu^*(p^{u-i})$ ($i = 1, \cdots, u$). Thus the right members in (5) and (6) are equal, so that $\mu_P(p^u) = \mu^*(p^u)$.

In view of the above result, it suffices to show that any function μ^* satisfying (4) is multiplicative, thus proving the following

Theorem 1. *If μ^* satisfies (4) for every $n \in Z$, then $\mu^* = \mu_P$.*

In order to prove this theorem we begin with some lemmata. ([2])

Lemma 1. *Let g be a multiplicative function. If $g(1) = 0 \Rightarrow g(n) = 0$ for every $n \in Z$. If $g(1) \neq 0 \Rightarrow g(1) = 1$.*

Lemma 2. *Let f be an arithmetical function. If $\sum_{d|n, d \in Q} f(n/d) = 0$ for every $n \in Z$, then $f(n) = 0$ for every $n \in Z$.*

Proof. As in [1, lemma 2], we proceed by induction on n, noting that $f(n)$ always appears in the involved sum, because $1 \in Q$.

Lemma 3. *Let g be a multiplicative function. If f is such that*

$$g(n) = \sum_{\substack{d|n \\ d \in Q}} f\left(\frac{n}{d}\right)$$

then f is multiplicative.

Proof. The proof is a convenient and trivial adaptation of that of lemma 3, [2]. For the sake of clearness we repeat it. If $g(1) = 0$, then $g(n) = 0$ for every $n \in Z$, so by lemma 2, $f(n) = 0$ for every $n \in Z$. If $g(1) = 1$ it is clear that that $f(1) = 1$. Let us consider the following proposition:

$$P \wr m, n \wr : \text{“}(m, n) = 1 \Rightarrow f(mn) = f(m)f(n)\text{”} \ .$$

If m or $n = 1$, the above proposition is true. Let us suppose that $P \wr m, n \wr$ is false for some pair (m, n), and let

$$m_0 = \ \min \ \{ \ m \ ; \exists n \in Z \text{ with } h(m, n) = 1 \text{ such that } P \wr m, n \wr \text{ is false}\}$$

then there exists an n such that $(m_0, n) = 1$ and $P \wr m_0, n \wr$ is false. Now let

$$n_0 = \min\{n \ ; \ (m_0, n) = 1 \text{ and } P \wr m_0, n \wr \text{ is false } \}.$$

We have then :

i) $1 < m_0 < n_0$ and $(m_0, n_0) = 1$
ii) $P \wr m_0, n_0 \wr$ is false.
iii) $P \wr k, n \wr$ is true for every n and each k such that $1 \leq k < m_0$ and $(k, n) = 1$.
iv) $P \wr m_0, t \wr$ is true for each t such that $1 \leq t \leq n_0$ and $(m_0, t) = 1$

If now we take $g(m_0, n_0)$ we find

$$\begin{aligned} \sum_{\substack{t \mid m_0 n_0 \\ t \in Q}} &= g(m_0 n_0) = g(m_0) g(n_0) \\ &= \left\{ \sum_{\substack{d \mid m_0 \\ d \in Q}} f\left(\frac{m_0}{d}\right) \right\} \times \left\{ \sum_{\substack{\delta \mid m_0 \\ \delta \in Q}} f\left(\frac{m_0}{\delta}\right) \right\} \end{aligned}$$

using the multiplicativity of g and (1); so that

$$\sum_{\substack{d \mid m_0, \ \delta \mid n_0 \\ d, \delta \in Q}} \left\{ f\left(\frac{m_0 n_0}{d\delta}\right) - f\left(\frac{m_0}{d}\right) f\left(\frac{n_0}{\delta}\right) \right\} = 0 \ .$$

But m_0/d and $_0/\delta$ smaller than m_0 and n_0, resp., if $d, \delta \neq 1$; thus from the above relation and the hypothesis on (m_0, n_0), we conclude that

$$f(m_0 n_0) - f(m_0) f(n_0) = 0,$$

contradicting ii). So $P \wr m, n \wr$ is always true.

We remark that no explicit calculation for μ_P is needed in the above reasoning. Further, if we use the following theorem [1, theorem 3] :

Theorem 2. *If f and g are arithmetical functions then*

$$\text{(iii)} \qquad g(n) = \sum_{\substack{d \mid n \\ d \in Q}} f\left(\frac{n}{d}\right) \Leftrightarrow f(n) = \sum_{d \mid n} g(d) \mu_P \left(\frac{n}{d}\right),$$

the uniqueness of μ_P is easily proved. Here we have proved Theorem 1 without help of this result; but we will prove more: μ_P is the sole function that can perform the inversion in Theorem 2. For this we have to prove the following

Lemma 4. *If $f(1) \neq 0$ and*

$$\sum_{e|n} f(e)\rho^*\left(\frac{n}{e}\right) = f(n),$$

then $\rho^*(n) = \rho(n)$ for every $n \in Z$.

Proof. See [2, lemma 4].

Suppose now that μ^* is such that

$$g(n) = \sum_{\substack{d|n \\ d\in Q}} f\left(\frac{n}{d}\right) \Leftrightarrow f(n) = \sum_{d|n} g(d)\mu^*\left(\frac{n}{d}\right). \tag{7}$$

Then

$$\begin{aligned} f(n) &= \sum_{d|n} g(d)\mu^*\left(\frac{n}{d}\right) = \sum_{d|n} \mu^{ast}\left(\frac{n}{d}\right) \sum\nolimits_{\substack{\delta e=d \\ \delta\in Q}} f(e) \\ &= \sum_{e|n} f(e) \sum\nolimits_{\substack{d\delta'=n \\ \delta e=d \\ \delta\in Q}} \mu^*(\delta') = \sum_{e|n} f(e) \sum\nolimits_{\substack{\delta\delta'=\frac{n}{e} \\ \delta\in Q}} \mu^*(\delta')\ ; \end{aligned}$$

writing $\rho^*(n) = \sum_{n=\delta d,\ d\in Q} \mu^*(\delta)$ we have $f(n) = \sum_{e|n} f(e)\rho^*(n/e)$, so by lemma 4 and theorem 1,

$$\rho(n) = \rho^*(n) = \sum \mu^*(\delta) \Rightarrow \mu^* = \mu_P.$$

Thus we have proved the

Theorem 3. *Let f and g be arithmetical functions such that $g(1) \neq 0$. If*

$$g(n) = \sum_{\substack{d|n \\ d\in Q}} f\left(\frac{n}{d}\right) \Leftrightarrow f(n) = \sum_{d|n} g(d)\mu^*\left(\frac{n}{d}\right)$$

then $\mu^* = \mu_P$.

References

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Recibido en marzo de 1968

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