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RINGS OF FRACTIONS OF EUCLIDEAN RINGS[1]

Victor S. Albis-González

Raj K. Markanda

Departamento de Matemáticas y Estadistica
Universidad Nacional de Colombia
Bogotá, 6, D.E., Colombia

1. **Introduction**. In this paper we discuss the structure of rings of fractions of left Euclidean rings and left Bézout rings. Analogous results hold for right Euclidean rings and right Bézout rings. For the sake of completeness we give the basic definitions.

We say that a ring A with unity is *left Euclidean* if there exists an ordinal-valued function $\phi : A \to W$ such that given a, b in A, $\neq 0$, there exists q, r in A such that $a = qb + r$ and $\phi(r) < \phi(b)$. On the other hand, we say that a ring A with unity is *left Bézout* if every finitely generated left ideal of A is left principal. *Mutatis mutandi*, we have the corresponding *right* notions.

In Theorem 1 (resp., Theorem 2) we show that if A is left Euclidean (resp., left Bézout) and B is an overring of A such that given x in B, ux is in A for some left unit of B, then B is left Euclidean (resp., left Bézout). In Theorem 3 we show that a ring B such that $A \subseteq B \subseteq S^{-1}A$ has the same structure as A, where $S^{-1}A$ is the right ring of fractions of A.

In Section 2 we shall recall some facts about rings of fractions in the non-connnutative case, which are not easy to find in the literature with complete details. Finally, in Section 3, we prove the main results.

2. **Rings of fractions in the non-commutative case**. In this Section we discuss some basic facts about rings of fractions in the non-commutative

case, skipping however the details. See [**2**: Chap. II, 2, exc. 22-23, pp. 132-3] and [**3**; p. 108].

Let A be a ring (not necessarily commutative) and S be a multiplicative subset of A. An ordered pair (B, ι), where B is a ring and $\iota : A \to B$ is a homomorphism, is called a *ring of right fractions of* A, with denominators in S, if it satisfies the following conditions:

F_1 . If $\iota(a) = 0$, there exists $s \in S$ such that $as = 0$.
F_2 . If $s \in S$, $\iota(s)$ is invertible in B.
F_3 . Every element of B is of the form $\iota(a)(\iota(s))^{-1}$.

Here we shall only consider the particular case when S consists of regular elements of A, the general case being always reducible to this situation. Under this hypothesis, F_1 means that i is a monomorphism, enabling us consequently to identify a with its image $\iota(a)$ in B.

It can be shown [l; p. 132] that the following is a necessary and sufficient condition for the existence of a right ring of fractions:

C. For every $(a, s) \in A \times S$ there exists $(a', s') \in A \times S$ such that $as' = sa'$.

Moreover, whenever A and S satisfy condition C, a pair (B, ι) satisfying F_1, F_2, F_3 can be constructed as follows ; In $A \times S$ define an equivalence relation R thus:

$$(a, s)R(b, t) \quad \text{if there exist} \quad u, v \in A$$

$$\text{such that} \quad su = tv \in S \quad \text{and} \quad au = bv.$$

The image of (a, s) under the canonical map from $A \times S$ onto $(A \times S)/R$ will be denoted by as^{-1}. Now define the following operations on $(A \times S)/R$:

$$as^{-1} + bt^{-1} = (av + bu)\sigma^{-1}$$

where $u, v \in A$ and $tu = sv = \sigma \in S$ (remark that σ plays the role of a common denominator; the existence of such $u, v \in A$ follows using condition C for $(s, t) \in S \times S$), and

$$(as^{-1})(bt^{-1}) = (au)(tv)^{-1} \ ,$$

where $u, v \in A$ and $bv == su$, $v \in S$ (again the existence of such u and v follows from C applied to $(b, s) \in A \times S$). These (well-defined) operations make $(A \times S)/R$ into a ring with unity: moreover, the map $\iota : a \mapsto a1^{-1})$ is monomorphism from A into $(A \times S)/R$ and it is not difficult to show that $((A \times S)/R, \iota)$ satisfies F_1, F_2, F_3.

On the other hand, if (B, i) is a right ring of fractions with denominators in S, this ring has the following universal property: *for every ring homomorphism* $f : A \to C$ *such that* $f(s)$ *is invertible in* C *for all* $s \in S$ *there exists a unique homomorphism* $g : B \to C$ *such that* $f = g \circ \iota$ [2; exc. 22 (c), p.133]. This result allows us to speak of the right ring of fractions of A with denominators in S; from now on this ring will be denoted by the familiar notation $(S^{-1}A, \iota)$.

As in the commutative case, we say that a multiplicative subset of a ring A is *satured* if $xy \in S$ implies $x \in S$ and $y \in S$.

A very important case occurs when A has no divisors of zero, so that $S = A \setminus \{0\}$ is a multiplicative subset of A. If in this case $S^{-1}A$ exists it is called *the total right ring of fractions of* A. To this respect the following proposition holds:

PROPOSITION 1. ([**3**; p. 109], [**2**; exc. 23, p. 133]). *Let A be a ring with no divisor of zero. Then for $S = A \setminus \{0\}$, $S^{-1}A$ exists if, and only if, the following condition holds:*

O. For every $a, b \in A$, $b \neq =$, there exist $u, v \in A$ such that $au = bv \neq 0$.

Moreover, any right noetherian ring with no divisor of zero satisfies condition O.

A ring with no divisor of zero which satisfies condition O above is called a *right Ore ring*.

The notion of a ring of left fractions of A with denominators in S is similarly defined and the above theory carries over with the adequated changes.

3. **The main results**. Our first Theorem extends Proposition 7 of [**5**] to the non-commutative case and at the same time shows that the condition of non-existence of divisors of zero is not essential in the said proposition.

THEOREM 1. *Let A be a left Euclidean ring and let B be an overring of A. If for every x in B there exists a left unit u of B such that ux is in A, then B is left Euclidean.*

Proof. Let ϕ be a left algorithm on A and for every $x \in B$ define

$$\theta(x) = \min\{\phi(ux); ux \in A \text{ for some left unit } u \text{ of } B\}.$$

We claim that θ is a left algorithm on B. Indeed, let x, y be in B, $y \neq 0$; then, by hypothesis, there are left units u, v of B such that $a = ux$, $b = vy$ are in A. We may choose v such that $\theta(y) = \phi(vy) = \phi(b)$. Since A is left Euclidean, there exist q, r in A such that

$$a = qb + r \quad \text{with} \quad \phi(r) < \phi(b),$$

i.e.,

$$x = (u^{-l}qv)y + u^{-1}r \quad \text{with} \quad \theta(u^{-l}r) < \theta(y).$$

COROLLARY 1.1. *Let S be a multiplicative subset of a left Euclidean ring. If S consists of regular elements of A and satisfies condition C, then $S^{-1}A$, the left ring of fractions of A, with denominators in S, exists and is left Euclidean.*

Proof. Take $B = S^{-1}A$ and note that $s(s^{-1}x) = x$ in A for every element $s^{-1}a$ of B.

COROLLARY 1.2. *If A is a commutative ring and S is a multiplicative subset of A, then $S^{-1}A$ is Euclidean whenever A is Euclidean.*

Proof. Let $\iota : A \to S^{-1}A$ be the canonical homomorphism; then $\iota(A)$ is Euclidean [**4**; Prop. 1.9] whence $S^{-1}A$ s Euclidean.

Theorem 1 can be generalized to left Bézout rings by noting that every left ideal in B is an extension of a left ideal of A. Thus we have the following

THEOREM 2. *Let A be a left Bézout ring and let B be an overring of A satisfying the conditions of Theorem 1. Then B is left Bézout.*

Using the fact that two non-zero elements of a left Bézout ring have a right g.c.d., we get the following

THEOREM 3. *Let A be a left Bézout ring and S a saturated multiplicative subset of A for which the right ring of fractions exists. If B is an overring of A in $S^{-1}A$ (i.e., $A \subseteq B \subseteq S^{-1}A$) then B is of the form $T^{-1}A$ for some multiplicative subset $T \subseteq S$ and thus left Bézout.*

Proof. Let $T = \{s \in S; s^{-1} \in B\}$ which clearly is a multiplicative subset of A. We claim that $B = T^{-1}A$. Indeed, since S is saturated every element $as^{-1} \in B$ is of the form $as^{-1} = a_1 s_1^{-1}$, with right g.c.d.$(a_1, s_1) = 1$; this implies that $1 = ca_1 + ds_1$, for some c, d in A, and thus $s_1^{-1} = l.s_1^{-1} = c(a_1 s_1^{-1}) + d \in B$; therefore, if $\iota : A \to B$ denotes the inclusion, we see that every element of B can be written as $\iota(a_1)(\iota(s_1))^{-1}$, where $a_1 \in A$ and $s_1 \in T$; i.e., F_3 is satisfied. Finally F_2 is satisfied by the very definition of T, and this suffices to prove that (B, ι) is the ring of right fractions of A with denominators in T, since F_1 is trivially verified. By Theorem 2, B is left Bézout.

COROLLARY 3.1. *If A is left Euclidean and $S^{-1}A$ and B are as in Theorem 3, then B is left Euclidean and, moreover, of the form $T^{-1}A$, for some $T \subseteq S$.*

Finally we wish to close this section with the remark that the conditions of Theorem 3 actually hold, especially in the Euclidean case. In fact, we have constructed in [**l**] a vast class of two-sided Euclidean rings for which those conditions hold. In particular, since orders in a (finite dimensional) division algebra are two-sided Ore rings, by virtue of Proposition 1, we have

COROLLARY 3.2. *Let A be an order of a finite dimensional division algebra K over $\mathbb{Q}$ or $k[x]$, where k is a finite field. If A is left Euclidean, then any order B of K containing A is of the form $T^{-1}A$, where $T \subseteq A \setminus \{0\}$. Moreover, B is left Euclidean.*

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