ZETA FUNCTIONS OF SINGULAR CURVES OVER FINITE FIELDS VIA RIEMANN-ROCH THEOREM

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Abstract. Let X be a complete, geometrically irreducible, algebraic curve defined over a finite field $\mathbb{F}_q$, and $\zeta(X,t)$ its zeta function [Ser1]. If X is a singular curve, we have two other zeta functions. The first of them is the Dirichlet series $Z(Ca(X),t)$ associated to effective Cartier divisors on X, and the second one is the Dirichlet series $Z(Div(X),t)$ associated to effective divisors on X. In this paper we generalize F. K. Schmidt's results on the rationality and functional equation of the zeta function $\zeta(X,t)$ of a non-singular curve [Sch] to the functions $Z(Ca(X),t)$ and $Z(Div(X),t)$ using the singular Riemann-Roch theorem.

In [G] Galkin introduced a zeta function for orders in global fields, which coincides with Schmidt's zeta function in the case of a non-singular curve and satisfies a functional equation in the Gorenstein case only. The zeta function studied by Galkin coincides with the zeta function $Z(Div(X),t)$, if X is Gorenstein. Green [Gr] obtained a functional equation by modifying the definition of zeta function given by Galkin, however this zeta function is not uniquely determined by the curve. Stöhr [Sto] defined the zeta function $Z(Div(X),t)$, which coincides in the Gorenstein case with the zeta function of Galkin and always satisfies a functional equation. Our results complete those contained in [G],[Gr],[Sto] in the case of function fields. We follow an approach based in the singular Riemann-Roch theorem instead of an adelic approach a lá Tate used in [G] and [Gr]. This permits to find the residue at $s=0$ of the zeta function $Z(Ca(X),q^{-s})$. On the other hand, as far as the author knows the function $Z(Ca(X),t)$ does not appear repported in the literature before. In a forthcoming paper, we shall study from a geometrical point of view, the local factors of the zeta function $Z(Ca(X),t)$. I would like to thank s to Prof. Karl-Otto Stöhr for many useful conversations.

1. Preliminaries

In this section we present the basic facts about singular curves and zeta functions (cf. e.g. [Ros1], [Ros2], [Ser2], [Sto]). We use as reference for the results of this section Stöhr's paper [Sto].

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Let X be a complete, geometrically irreducible, algebraic curve defined over a field k. We denote by $K = k(X)$ the function field of X over k, g its arithmetic genus, $\tilde{g}$ its geometric genus, $\tilde{X}$ the normalization of X over k (also named the non-singular model of X) and $\pi : \tilde{X} \longrightarrow X$ the normalization map. The regular surjective map π is birational. In particular, the function field of the smooth curve $\tilde{X}$ is K/k.
By a divisor of X, we mean a coherent fractional ideal sheaf, or equivalently a formal product

$$\mathcal{A} = \prod_{P \in X} \mathcal{A}_P,$$

where the P-component $\mathcal{A}_P$ (i.e. the stalk of $\mathcal{A}$ at P) is a non-zero fractional ideal of $\mathcal{O}_P$ for each point P, and $\mathcal{A}_P = \mathcal{O}_P$ for all but finitely many points. Given two divisors $\mathcal{A}$ and $\mathcal{B}$, we define the product divisor $\mathcal{AB}$ and the quotient divisor $\mathcal{A} : \mathcal{B}$ pointwise by:

$$(\mathcal{AB})_P := \mathcal{A}_P \mathcal{B}_P,$$

$$(\mathcal{A} : \mathcal{B})_P := \{z \in K | z\mathcal{B}_P \subseteq \mathcal{A}_P\}.$$

We denote by $Div(X)$ the set of divisors of X. A divisor $\mathcal{A}$ is called a Cartier divisor or locally principal, if each $\mathcal{A}_P$ is a principal fractional ideal. The Cartier divisors form a multiplicative group, having the structure divisor

$$\mathcal{O} := \prod_{P \in X} \mathcal{O}_P,$$

as the identity. We denote the group of Cartier divisors by $Ca(X)$. We define a partial order on $Div(X)$, by

$$\mathcal{A} \leqslant \mathcal{B} \Longleftrightarrow \mathcal{A}_P \subseteq \mathcal{B}_P \quad \text{for all} \quad P \in X.$$

A divisor $\mathcal{A}$ is called effective if $\mathcal{A} \geq \mathcal{O}$. For our purposes, it is more convenient to work with the above ordering than with the usual one.
The degree of a divisor is uniquely defined by the following properties:
(i) deg($\mathcal{O}$)=0.
(ii) deg($\mathcal{B}$) - deg($\mathcal{A}$) $= \sum_{P \in X} dim_k(\mathcal{B}_P/\mathcal{A}_P)$, whenever $\mathcal{B} \geq \mathcal{A}$.
We note that $deg(\mathcal{AB}) \neq deg(\mathcal{A}) \;\; deg(\mathcal{B})$ (cf. [Ha], sect. 1). However, if at least one of divisors $\mathcal{A}$ or $\mathcal{B}$ is a Cartier divisor then the equality holds.
For each non-zero rational function $z \in K^*$, let $div(z)$ be its principal divisor, defined as

$$div(z) := \prod_{P\in X} z^{-1}\mathcal{O}_P.$$

We denote by $Prin(X)$ the subgroup of principal divisors of X.
Let

$$L(\mathcal{A}) := \bigcap_{P\in X} \mathcal{A}_P = \{z \in K | div(z)\mathcal{A} \geqq \mathcal{O}\}$$

be the $k-$vector space of global sections of $\mathcal{A}$ (also denoted by $H^0(X, \mathcal{A})$). We denote the dimension of the above $k-$vector space by $l(\mathcal{A})$ (also denoted $h^0(X, \mathcal{A})$). The Riemann-Roch theorem for function fields was generalized by Rosenlicht to curves with singularities (cf. e.g. [Ros1], [Sto]).

Theorem 1.1. (Riemann-Roch theorem for singular curves)
Each divisor $\mathcal{A}$ of X satisfies

$$l(\mathcal{A}) = deg(\mathcal{A}) + 1 - g + l(\mathcal{C} : \mathcal{A}),$$

where $\mathcal{C}$ denotes the canonical divisor of X.
The local duality theorem also generalizes to singular curves (cf. e.g. [Sto], thm. 1.5)

Theorem 1.2. (Local duality)
Let $\mathcal{A}$, $\mathcal{B}$ be divisors of X, such that $\mathcal{A} \geqq \mathcal{B}$. Then for each point P, we have the following $k-$isomorphism

$$(\mathcal{C}_P : \mathcal{B}_P)/(\mathcal{C}_P : \mathcal{A}_P) \xrightarrow[\cong]{} \mathrm{Hom}_k(\mathcal{A}_P/\mathcal{B}_P, k).$$

As a consequence of theorem of local duality, we get the reciprocity (cf. e.g. [Sto], 1.7)

Corollary 1.3. (Reciprocity)
For each divisor $\mathcal{A}$, we have

$$\mathcal{C} : (\mathcal{C} : \mathcal{A}) = \mathcal{A}.$$

From, now on, we understand point to mean closed point. Let P be a point of X and $\mathcal{O}_P$ the local ring of X at P. Let $Q_1, Q_2, .., Q_d$ be the points of $\tilde{X}$ lying over P, i.e. $\pi^{-1}(P) = \{Q_1, .., Q_d\}$, and $\mathcal{O}_{Q_1}, ..., \mathcal{O}_{Q_d}$ the corresponding local rings at these points. Since the function fields of X and $\tilde{X}$ are the same, and $\widetilde{X}$ is a non-singular curve, the local rings $\mathcal{O}_{Q_1}, ..., \mathcal{O}_{Q_d}$ are valuation rings of K/k lying over $\mathcal{O}_P$. The integral closure of $\mathcal{O}_P$ in K is $\widetilde{\mathcal{O}}_P = \bigcap_{Q\in\pi^{-1}(P)} \mathcal{O}_Q$.

The degree of singularity of X at P is defined as

$$\delta_P = dim_k(\tilde{\mathcal{O}}_P/\mathcal{O}_P).$$

By theorem. 1 of [Ros1], $\delta_P < \infty$. The degree of singularity of X is defined as

$$\delta = \sum_{P \in X} \delta_P.$$

The degree of singularity δ_P remains invariant under completion. The total degree of singularity δ remains invariant under separable constant extensions (cf. [Ros1], thm. 12).

We recall the genus formula of a complete, geometrically irreducible, algebraic curve X [H]:

$$g = \tilde{g} + \sum_{P \in X} \delta_P,$$

where $\tilde{g}$ is the geometric genus of X.

The conductor ideal $\mathcal{F}_P$ of $\widetilde{\mathcal{O}}_P$ in $\mathcal{O}_P$ is defined as

$$\mathcal{F}_P = \{x \in K | x\widetilde{\mathcal{O}}_P \subseteq \mathcal{O}_P\}.$$

This ideal is the largest common ideal of $\widetilde{\mathcal{O}}_P$ and $\mathcal{O}_P$. Furthermore $\mathcal{F}_P \neq 0$, since $\delta_P < \infty$. On the other hand, $\widetilde{\mathcal{O}}_P$ is a Dedekind domain with a finite number of maximal ideals, thus $\mathcal{F}_P$ is an $\widetilde{\mathcal{O}}_P$-principal ideal. The degree of the conductor ideal is defined as

$$deg\mathcal{F}_P := dim_k(\widetilde{\mathcal{O}}_P/\mathcal{F}_P).$$

The degree of $\mathcal{F}_P$ is also invariant under completions and under separable constant extensions.

We say that a local ring $\mathcal{O}_P$ is Gorenstein if and only if $deg\mathcal{F}_P = 2\delta_P$. An algebraic curve is called a Gorenstein if all its local rings are Gorenstein.

The dualizing sheaf ω_X is locally free of rank 1 if and only if X is a Gorenstein curve (cf. [A-K] chap. VIII, prop. 1.16). We denote by $\mathcal{C}$ the canonical divisor of a complete, geometrically irreducible, algebraic curve X. Then $\mathcal{C}_P$ is a principal ideal if and only if $\omega_{X,P}$ is free of rank 1. Summarizing, we have the following result due to Rosenlicht.

Theorem 1.4.

Let X be a complete, geometrically irreducible, algebraic curve defined over k. The curve X is Gorenstein if and only if its canonical divisor is a Cartier divisor.

We denote by $Ca^0(X)$ the subgroup of degree zero divisors of $Ca(X)$, by $Pic(X)$ and $Pic^0(X)$ the quotient groups $Ca(X)/Prin(X)$ and $Ca^0(X)/Prin(X)$, respectively.

If k is an algebraically closed field, $Pic^0(X)$ is the generalized Jacobian of X. By the approximation lemma, the above definition of the generalized Jacobian coincides with the definition given by Rosenlicht [Ros2]. The following sequence is exact (c.f. [Ha1], chap II, ex. 6.9)

$$1 \longrightarrow \prod_{P \in X_{sing}} \tilde{\mathcal{O}}_P^*/\mathcal{O}_P^* \longrightarrow Pic^0(X) \longrightarrow Pic^0(\tilde{X}) \longrightarrow 1. \tag{1.1}$$

If k is a finite field, $Pic^0(\tilde{X})$ is the group of divisor classes of degree zero of K. This group is finite (cf. [Sti], chap. V). We will see later that if k is a finite field then the group $\prod_{P \in X_{sing}} \tilde{\mathcal{O}}_P^*/\mathcal{O}_P^*$ is also finite group. Thus, it will follow from (1.1) that $Pic^0(X)$ is a finite group.

On the other hand, we have also the following zeta function

$$Z(Ca(X),t) = \sum_{\mathcal{A} \geq \mathcal{O}} t^{deg(\mathcal{A})}, \tag{1.2}$$

where $\mathcal{A}$ runs through all effective Cartier divisors on X. We shall see later on that $Z(Ca(X),t)$ converges analytically and uniformly on the semiplane $Re(s) > 1$. This zeta function decomposes formally into an Euler product

$$Z(Ca(X),t) = \prod_{P \in X} Z(\mathcal{O}_P, t), \tag{1.3}$$

where

$$Z(\mathcal{O}_P, t) = \sum_{I \supseteqq \mathcal{O}_P} t^{dim_k(I/\mathcal{O}_P)},$$

and where I runs through all fractional principal ideals of $\mathcal{O}_P$, such that $I \supseteqq \mathcal{O}_P$. We note that each ideal I is expressed as $I = f_P^{-1}\mathcal{O}_P$, $f_P \in \mathcal{O}_P \setminus \{0\}$. The correspondence $I \longrightarrow I^{-1}$ is a bijection between the principal fractional ideals containing $\mathcal{O}_P$ and the principal ideals contained in $\mathcal{O}_P$. As it preserves degrees, we have

$$Z(\mathcal{O}_P, t) = \sum_{I \subseteq \mathcal{O}_P} t^{\dim_k(\mathcal{O}_P/I)}.$$

The product (1.3) converges absolutely on the semiplane $Re(s) > 1$. In fact, Since we can remove a finite number of points in the product (1.4) without affecting its convergence, the result follos from the fact that $\zeta(X,t)$ has a convergenr Euler product in the semiplane $Re(s) > 1$ (cf. [Sti], chap. V, prop. 1.8).

2. Rationality and Functional Equation

In this section we present a generalization of results of F. K. Schmidt on the rationality and on the functional equation of $\zeta(X,t)$ to the function $Z(Ca(X),t)$ (cf. [Sti], chap. V). We also obtain similar results for the zeta function

$$Z(Div(X),t)=\sum_{\mathcal{A}\geq\mathcal{O}} t^{deg(\mathcal{A})},$$

where $\mathcal{A}$ runs through the effective divisors of X.

Lemma 2.1.
Let X be a complete, geometrically irreducible algebraic curve over a finite field $k=\mathbb{F}_q$. Then the zeta function $Z(Ca(X),t)$ is a rational function. More precisely,

$$Z(Ca(X),t)=\frac{L(Ca(X),t)}{(1-t)(1-qt)},$$

where $L(Ca(X),t)\in\mathbb{Z}[t]$ is a polynomial of degree at most $2g$, and $L(Ca(X),1)=\#Pic^0(X)$.

Proof.
In order to generalize the proof of F.K. Schmidt we need to prove the following three claims.
Claim 1

$$\#Pic^0(X)<\infty. \tag{2.1}$$

Claim 2
There exists a Cartier divisor of degree 1.
Claim 3
For any integer d, the number of divisor classes in $Pic(X)$ of degree d is independent of d and is equal to the cardinality of $Pic^0(X)$.
After this, we can follow the argument of F.K. Schmidt as in [Sti] (chap. V). It is important to note that in this argument it is irrelevant to know whether a canonical divisor is a Cartier divisor or not.
To establish claim 1, it is sufficient to show that the kernel of the morphism

$$\pi_0^*: Pic^0(X)\longrightarrow Pic^0(\tilde{X})$$

is a finite group. Denote by $[\mathcal{A}]$ the linear equivalence class of a divisor $\mathcal{A}$ and $\pi^*(\mathcal{A})$ the pullback of $\mathcal{A}$. Thus

$$\pi_0^*([\mathcal{A}])=[\pi^*(\mathcal{A})].$$

We note that $[\mathcal{A}] \in ker(\pi_0^*)$ if and only if $\pi^*(\mathcal{A})$ is a principal divisor of $\tilde{X}$, i.e.

$$\pi^*(\mathcal{A})_Q = z^{-1}\mathcal{O}_Q = \mathcal{A}_P\mathcal{O}_Q, \tag{2.2}$$

for some $z \in K^*$ and every point Q of $\widetilde{X}$ lying over P. Therefore

$$\mathcal{F}_P \subseteqq z\mathcal{A}_P \subseteqq \tilde{\mathcal{O}}_P.$$

Since $deg\mathcal{F}_P < \infty$, the above relation implies that the kernel of π_0^* contains only a finite number of linear equivalence classes. The equivalence class of an effective Cartier divisor $\mathcal{A}$ contains $\frac{q^{l(\mathcal{A})-1}}{q-1}$ linearly equivalent Cartier divisors. Therefore the kernel of π_0^* is a finite group.

The second claim can be reduced to the non-singular case using the fact that the morphism $\pi^* : Ca(X) \longrightarrow Ca(\tilde{X})$ is surjective and preserves degrees. Thus, there exists a Cartier divisor of degree 1.

We observe that claim 1 and claim 2 imply that the number of effective Cartier divisors with a given degree is finite. Using the same argument as in the non-singular case, we prove that $Z(Ca(X), t)$ converges absolutely and uniformly on the semiplane $Re(s) > 1$ (cf. [Sti], chap. V, prop. 1.6).

The last claim follows from claim 2 and the fact that $deg(\mathcal{A}\mathcal{B}) = deg(\mathcal{A}) + deg(\mathcal{B})$, for any two Cartier divisors $\mathcal{A}$ and $\mathcal{B}$. □

Corollary 2.2.

$\frac{Z(\mathcal{O}_P,t)}{Z(\widetilde{\mathcal{O}}_P,t)}$ is a rational function.

Proof.

Taking a partial resolution of singularities of X we can suppose that P is the only singular point of X. Then

$$\frac{Z(Ca(X), t)}{Z(Ca(\tilde{X}), t)} = \prod_{Q\in\pi^{-1}(P)} (1 - t^{deg(Q)})\ Z(\mathcal{O}_P, t) = \frac{Z(\mathcal{O}_P, t)}{Z(\widetilde{\mathcal{O}}_P, t)}$$

The result follows from the previous theorem. □

Theorem 2.3.

Let X be a complete, geometrically irreducible, algebraic curve defined over a finite field $k = \mathbb{F}_q$. Then the zeta function $Z(Ca(X), t)$ satisfies the functional equation

$$Z(Ca(X), t) = q^{g-1}t^{2g-2}Z(Ca(X), \frac{1}{qt}), \tag{2.3}$$

if and only if X is a Gorenstein curve.

Proof.

If X is a Gorenstein curve, the argument of Schmidt for the non-singular case, the reciprocity (cf. cor. 1.2) and the observations made in the proof of lemma 2.1 imply the functional equation (cf. [Sti] chap. V, prop. 1.13). Conversely, if the zeta function $Z(Ca(X),t)$ satisfies the functional equation (2.3), and $g \geqq 1$, then the Riemann-Roch theorem and (2.1) imply

$$Z(Ca(X),t) = Z_1(X,t) + Z_2(X,t),$$

with

$$Z_1(X,t) = \frac{1}{q-1} \sum_{0 \leq deg[\mathcal{A}] \leq 2g-2} q^{l(\mathcal{A})} t^{deg(\mathcal{A})} = \frac{1}{q-1} \sum_{j=0}^{2g-2} a_j t^j, \tag{2.4}$$

and

$$Z_2(X,t) = \frac{1}{q-1} \sum_{deg[\mathcal{A}] \geqq 2g-1} q^{l(\mathcal{A})} t^{deg(\mathcal{A})} - \frac{1}{q-1} \sum_{deg[\mathcal{A}] \geqq 0} t^{deg(\mathcal{A})}$$

$$= \frac{1}{q-1} \left(\# Pic^0(X) q^g t^{2g-1} \frac{1}{1-qt} - \frac{\# Pic^0(X)}{1-t} \right). \tag{2.5}$$

By hypothesis $Z(Ca(X),t)$ satisfies the functional equation (2.3) and one verifies directly that the function $Z_2(X,t)$ also satisfies the functional equation (2.3). Therefore $Z_1(X,t)$ satisfies the functional equation (2.3). This implies that

$$a_{2g-2-j} = a_j q^{g-1-j}, \qquad j = 0, 1, .., 2g-2.$$

On the other hand, $a_0 = 1$ thus $a_{2g-2} = q^{g-1}$. Since $g \geq 1$, a divisor class $[\mathcal{C}]$ with deg($\mathcal{C}$)=$2g-2$ and $l(\mathcal{C}) = g$, appears in the sum in (2.4). These properties characterize the canonical class. Hence, by theorem 1.4, X is Gorenstein. In the case $g = 0$, the genus formula implies $\delta = 0$, so X is a non-singular curve. Thus in this case also X is Gorenstein. □

As a consequence of the functional equation (2.3), the degree of the polynomial $L(Ca(X),t)$ is $2g$.

Corollary 2.4.

The zeta function $Z(\mathcal{O}_P,t)$ satisfies the functional equation

$$\frac{Z(\mathcal{O}_P,t)}{Z(\widetilde{\mathcal{O}}_P,t)} = q^{\delta_P} t^{2\delta_P} \frac{Z(\mathcal{O}_P,\frac{1}{qt})}{Z(\widetilde{\mathcal{O}}_P,\frac{1}{qt})}$$

if and only if $\mathcal{O}_P$ is Gorenstein.

Remark 2.5.
The zeta function $Z(Div(X),t)$, associated to the set of divisors $Div(X)$, is a rational function. More precisely,

$$Z(Div(X),t)=\sum_{\mathcal{A}\geq\mathcal{O}} t^{deg(\mathcal{A})}=\frac{L(Div(X),t)}{(1-t)(1-qt)},$$

where $\mathcal{A}$ runs through all divisors of X. $L(Div(X),t)$ is a polynomial in $\mathbb{Z}[t]$ of degree at most $2g$ and $L(Div(X),1)=\#Cl^0(X)$, the number of classes of divisors of degree zero.
The proof is similar to the proof of lemma 2.1. In order to prove that

$$\#Cl^0(X)=Div^0(X)/Prin(X)<\infty,$$

we observe that $Pic^0(X)$ acts on $Cl^0(X)$ by multiplication and the quotient set $Cl^0(X)/Pic^0(X)$ is isomorphic to

$$\{\prod_{P\in X_{sing}} \mathcal{A}_P|\mathcal{A}_P \text{ is an ideal in } \mathcal{O}_P \text{ and } \mathcal{F}_P\subseteq\mathcal{A}_P\subseteq\widetilde{\mathcal{O}}_P\}. \tag{2.6}$$

Since $\dim_k(\tilde{\mathcal{O}}_P/\mathcal{F}_P)<\infty$ and $\#Pic^0(X)<\infty$ (cf. (2.1)), we conclude from (2.6) that $\#Cl^0(X)<\infty$. Now we can follow the proof of lemma 2.1. The zeta function $Z(Div(X),t)$ satisfies the functional equation (2.3). Stöhr also proved that $Z(Div(X),t)$ satisfies the functional equation (2.3) (cf. [Sto], (4.7)).
The local factors of $Z(Div(X),t)$ are rational functions (as in cor. 2.2) and satisfy a functional equation (as in cor. 2.4). The functional equation implies that the degree of the polynomial $L(Div(X),t)$ is $2g$.

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