

TABLESON OPERATORS, p -ADIC PARABOLIC EQUATIONS AND ULTRAMETRIC DIFFUSION.

J. J. RODRÍGUEZ-VEGA AND W. A. ZÚÑIGA-GALINDO

ABSTRACT. We give a multidimensional version of the p -adic heat equation, and show that its fundamental solution is the transition density of a Markov process.

1. INTRODUCTION.

In recent years p -adic analysis has received a lot of attention due to its applications in mathematical physics, see e.g. [1], [2], [4], [5], [16], [17], [18], [22], [28] and references therein. One motivation comes from statistical physics, in particular in connection with models describing relaxation in glasses, macromolecules, and proteins. It has been proposed that the non exponential nature of those relaxations is a consequence of a hierarchical structure of the state space which can in turn be put in connection with p -adic structures ([4], [5], [22]). In [4] was demonstrated that the p -adic analysis is a natural basis for the construction of a wide variety of models of ultrametric diffusion constrained by hierarchical energy landscapes. To each of these models is associated a stochastic equation (the master equation). In several cases this equation is a p -adic parabolic equation of type:

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} + a(Au)(x,t) = f(x,t), & x \in \mathbb{Q}_p^n, \quad t \in (0, T], \\ u(x, 0) = \varphi(x), \end{cases} \quad (1.1)$$

where a is a positive constant, A is pseudo-differential operator, and \mathbb{Q}_p is the field of p -adic numbers. The simplest case occurs when $n = 1$ and A is the Vladimirov operator:

$$(D^\alpha \varphi)(x) = \mathcal{F}_{\xi \rightarrow x}^{-1} \left(|\xi|_p^\alpha \mathcal{F}_{x \rightarrow \xi} \varphi(x) \right), \quad \alpha > 0,$$

where \mathcal{F} is the Fourier transform. The fundamental solution of (1.1) is density transition of a time- and space-homogeneous Markov process, that is consider the p -adic counterpart of the Brownian motion (see [18], [28]).

It is relevant to mention that in the case $n = 1$, the fundamental solution of (1.1) when $A = D^\alpha$ (also called the p -adic heat kernel) has been studied extensively, see e.g. [6], [11], [12], [14], [18], [28].

A natural problem is to study the initial value problem (1.1) in the n -dimensional case. Recently, the second author considered Cauchy's problem (1.1) when

$$(A\varphi)(x) = \mathcal{F}_{\xi \rightarrow x}^{-1} \left(|f(\xi)|_p^\alpha \mathcal{F}_{x \rightarrow \xi} \varphi(x) \right), \quad \alpha > 0,$$

2000 *Mathematics Subject Classification.* Primary 35R60, 60J25; Secondary 47S10, 35S99.

Key words and phrases. Parabolic equations, Markov processes, p -adic numbers, ultrametric diffusion.

here $f(\xi)$ is an elliptic homogeneous polynomial in n variables, and the datum φ is a locally constant and integrable function. Under these hypotheses it was established the existence of a unique solution to Cauchy's problem (1.1). In addition, the fundamental solution is a transition density of a Markov process with space state \mathbb{Q}_p^n (see [29]).

In this paper we study Cauchy's problem (1.1) when A is the Taibleson pseudo-differential operator which is defined as follows:

$$\left(D_T^\beta \varphi\right)(x) = \mathcal{F}_{\xi \rightarrow x}^{-1} \left(\left(\max_{1 \leq i \leq n} |\xi_i|_p \right)^\beta \mathcal{F}_{x \rightarrow \xi} \varphi(x) \right), \quad \beta > 0. \quad (1.2)$$

Recently Albeverio, Khrennikov, and Shelkovich studied D_T^β in the context of the Lizorkin spaces [3].

We prove existence and uniqueness of the Cauchy problem (1.1-1.2) in spaces of increasing functions introduced by Kochubei in [19], see Theorem 1. We also associate a Markov processes to equation the fundamental solution (see Theorem 2). These results constitute an extension of the corresponding results in [18], [28].

Let us explain the connection between the results of this paper and those of [29]. There are infinitely many homogeneous polynomial functions satisfying

$$|f(\xi)|_p = \left(\max_{1 \leq i \leq n} |\xi_i|_p \right)^d, \quad \text{for any } \xi \in \mathbb{Q}_p^n,$$

here d denotes degree of f (c.f. Lemmas 14-15). Hence the pseudo-differential operators considered here are a subclass of the ones considered in [29]. However, the function spaces for the solutions and initial data are completely different. In this paper the initial datum and the solution to Cauchy problem (1.1-1.2) are not necessarily bounded, neither integrable, but in [29] are.

Finally, our results can be extended to operators of the form

$$(A\varphi)(x) = a_0(x, t)(D_T^\alpha \varphi)(x) + \sum_{k=1}^n a_k(x, t)(D_T^{\alpha_k} \varphi)(x) + b(x, t)\varphi(x), \quad (1.3)$$

$\alpha > 1$, $0 < \alpha_1 < \dots < \alpha_n < \alpha$; where the $a_k(x, t)$, and $b(x, t)$ are bounded continuous functions, using the techniques presented in [18]-[20]. These results will appear later elsewhere.

2. PRELIMINARY RESULTS

As general reference for p -adic analysis we refer the reader to [25] and [28]. The field of p -adic numbers \mathbb{Q}_p is defined as the completion of the field of rational numbers \mathbb{Q} with respect to the non-Archimedean p -adic norm $|\cdot|_p$ which is defined as follows: $|0|_p = 0$; if $x \in \mathbb{Q}^\times$, $x = p^\gamma \frac{a}{b}$ with a, b integers coprime to p , then $|x|_p = p^{-\gamma}$. The integer $\gamma = \gamma(x)$ is called the p -adic order of x , and it will be denoted as $ord(x)$. We use the same symbol, $|\cdot|_p$, for the p -adic norm on \mathbb{Q}_p . We extend the p -adic norm to \mathbb{Q}_p^n as follows:

$$\|x\|_p := \max_{1 \leq i \leq n} |x_i|_p, \quad \text{for } x = (x_1, \dots, x_n) \in \mathbb{Q}_p^n.$$

Note that $\|x\|_p = p^{-\min_{1 \leq i \leq n} \{ord(x_i)\}}$.

Any p -adic number $x \neq 0$ has a unique expansion of the form

$$x = p^\gamma \sum_{j=0}^{\infty} x_j p^j,$$

where $\gamma = \text{ord}(x) \in \mathbb{Z}$, and $x_j \in \{0, 1, \dots, p-1\}$. By using the above expansion, we define the *fractional part of $x \in \mathbb{Q}_p$* , denoted as $\{x\}_p$, as the following rational number:

$$\{x\}_p := \begin{cases} 0, & \text{if } x = 0, \text{ or } \gamma \geq 0 \\ p^\gamma \sum_{j=0}^{|\gamma|-1} x_j p^j, & \text{if } \gamma < 0. \end{cases}$$

Denote by $B_\gamma^n(a) = \{x \in \mathbb{Q}_p^n \mid \|x - a\|_p \leq p^\gamma\}$, the ball of radius p^γ with center at $a = (a_1, \dots, a_n) \in \mathbb{Q}_p^n$, and $B_\gamma^n(0) = B_\gamma^n$, $\gamma \in \mathbb{Z}$. Note that $B_\gamma^n(a) = B_\gamma(a_1) \times \dots \times B_\gamma(a_n)$, where $B_\gamma(a_j) = \{x_j \in \mathbb{Q}_p \mid |x_j - a_j|_p \leq p^\gamma\}$ is the one-dimensional ball of radius p^γ with center at $a_j \in \mathbb{Q}_p$. The Ball B_0^n equals the product of n copies of $B_0(0) = \mathbb{Z}_p$, the ring of p -adic integers.

Let $d^n x$ denote the Haar measure on \mathbb{Q}_p^n normalized by the condition $\int_{B_0^n} d^n x = 1$.

A complex-valued function φ defined on \mathbb{Q}_p^n is called *locally constant* if for any $x \in \mathbb{Q}_p^n$ there exists an integer $l(x) \in \mathbb{Z}$ such that $\varphi(x + x') = \varphi(x)$, for $x' \in B_{l(x)}^n$.

A function $\varphi : \mathbb{Q}_p^n \rightarrow \mathbb{C}$ is called *Schwartz-Bruhat function*, or *test function*, if it is locally constant with compact support. The \mathbb{C} -vector space of the Schwartz-Bruhat functions is denoted by $S(\mathbb{Q}_p^n)$. If $\varphi \in S(\mathbb{Q}_p^n)$, there exist an integer $l \geq 0$ such that $\varphi(x + x') = \varphi(x)$, for $x' \in B_{-l}^n$, and $x \in \mathbb{Q}_p^n$ (see e.g. [28, VI.1, Lemma 1]). The largest of such numbers $l = l(\varphi)$ is called *the exponent of local constancy of φ* .

Let $S'(\mathbb{Q}_p^n)$ denote the set of all functionals (distributions) on $S(\mathbb{Q}_p^n)$. All the functionals on $S(\mathbb{Q}_p^n)$ are continuous (see e.g. [28, VI.3]).

Given $\xi = (\xi_1, \dots, \xi_n)$, $x = (x_1, \dots, x_n) \in \mathbb{Q}_p^n$, we set $\xi \cdot x := \sum_{i=1}^n \xi_i x_i$. The Fourier transform of $\varphi \in S(\mathbb{Q}_p^n)$ is defined as

$$(\mathcal{F}\varphi)(\xi) = \int_{\mathbb{Q}_p^n} \Psi(-\xi \cdot x) \varphi(x) d^n x, \quad \xi \in \mathbb{Q}_p^n,$$

where $\Psi(-\xi \cdot x) = \prod_{i=1}^n \Psi(-\xi_i x_i) = \exp\left(2\pi i \sum_{i=1}^n \{-\xi_i x_i\}_p\right)$. The function $\Psi(\alpha x_j) = \exp\left(2\pi i \sum_{i=1}^n \{\alpha x_j\}_p\right)$ is called *the standard additive character of \mathbb{Q}_p* . The Fourier Transform is a linear isomorphism from $S(\mathbb{Q}_p^n)$ onto itself.

2.1. The Taibleson Operator. We set

$$\Gamma_p^{(n)}(\alpha) := \frac{1 - p^{\alpha-n}}{1 - p^{-\alpha}}, \quad \alpha \neq 0.$$

This function is called the *p -adic Gamma function*. The function

$$k_\alpha(x) = \frac{\|x\|_p^{\alpha-n}}{\Gamma_p^{(n)}(\alpha)}, \quad \alpha \in \mathbb{R} \setminus \{0, n\}, \quad x \in \mathbb{Q}_p^n,$$

is called *the multi-dimensional Riesz Kernel*; it determines a distribution on $S(\mathbb{Q}_p^n)$ as follows. If $\alpha \neq 0$, n , and $\varphi \in S(\mathbb{Q}_p^n)$, then

$$\begin{aligned} \langle k_\alpha(x), \varphi(x) \rangle &= \frac{1-p^{-n}}{1-p^{\alpha-n}} \varphi(0) + \frac{1-p^{-\alpha}}{1-p^{\alpha-n}} \int_{\|x\|_p > 1} \|x\|_p^{\alpha-n} \varphi(x) d^n x \\ &+ \frac{1-p^{-\alpha}}{1-p^{\alpha-n}} \int_{\|x\|_p \leq 1} \|x\|_p^{\alpha-n} (\varphi(x) - \varphi(0)) d^n x. \end{aligned} \quad (2.1)$$

Then $k_\alpha \in S'(\mathbb{Q}_p^n)$, for $\mathbb{R} \setminus \{0, n\}$. In the case $\alpha = 0$, by passing to the limit in (2.1), we obtain

$$\langle k_0(x), \varphi(x) \rangle := \lim_{\alpha \rightarrow 0} \langle k_\alpha(x), \varphi(x) \rangle = \varphi(0),$$

i.e., $k_0(x) = \delta(x)$, the Dirac delta function, and therefore $k_\alpha \in S'(\mathbb{Q}_p^n)$, for $\mathbb{R} \setminus \{n\}$.

It follows from (2.1) that for $\alpha > 0$,

$$\langle k_{-\alpha}(x), \varphi(x) \rangle = \frac{1-p^\alpha}{1-p^{-\alpha-n}} \int_{\mathbb{Q}_p^n} \|x\|_p^{-\alpha-n} (\varphi(x) - \varphi(0)) d^n x. \quad (2.2)$$

Lemma 1 ([25, Chap. III, Theorem 4.5]). *As elements of $S'(\mathbb{Q}_p^n)$, $(\mathcal{F}k_\alpha)(x)$ equals $\|x\|_p^{-\alpha}$, $\alpha \neq n$.*

Definition 1. *The Taibleson pseudo-differential operator D_T^α , $\alpha > 0$, is defined as*

$$(D_T^\alpha \varphi)(x) = \mathcal{F}_{\xi \rightarrow x}^{-1} (\|\xi\|_p^\alpha \mathcal{F}_{x \rightarrow \xi} \varphi), \text{ for } \varphi \in S(\mathbb{Q}_p^n).$$

As a consequence of the previous lemma and (2.2), we have

$$\begin{aligned} (D_T^\alpha \varphi)(x) &= (k_{-\alpha} * \varphi)(x) = \\ &= \frac{1-p^\alpha}{1-p^{-\alpha-n}} \int_{\mathbb{Q}_p^n} \|y\|_p^{-\alpha-n} (\varphi(x-y) - \varphi(x)) d^n y. \end{aligned} \quad (2.3)$$

The right-hand side of (2.3) makes sense for a wider class of functions, for example, for locally constant functions $\varphi(x)$ satisfying

$$\int_{\|x\|_p \geq 1} \|x\|_p^{-\alpha-n} |\varphi(x)| d^n x < \infty.$$

3. THE p -ADIC HEAT EQUATION AND THE TAIBLESON OPERATOR

In this paper we consider the following Cauchy problem:

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} + a(D_T^\alpha u)(x,t) = f(x,t), & x \in \mathbb{Q}_p^n, \quad t \in (0, T], \\ u(x,0) = \varphi(x), \end{cases} \quad (3.1)$$

where $a > 0$, $\alpha > 0$ and D_T^α is the Taibleson operator. In this section we show that (3.1) is a multi-dimensional analog of the p -adic heat equation introduced in [28].

3.1. The Fundamental Solution. The *fundamental solution* for the Cauchy problem (3.1) is defined as

$$Z(x, t) := \int_{\mathbb{Q}_p^n} \Psi(x \cdot \xi) e^{-at\|\xi\|_p^\alpha} d^n \xi. \quad (3.2)$$

Lemma 2. *The fundamental solution has the following properties:*

- 1) $Z(x, t) = (1 - p^{-n})\|x\|_p^{-n} \sum_{k=0}^{\infty} p^{-kn} e^{-at(q^{-k}\|x\|_p^{-1})^\alpha} - \|x\|_p^{-n} e^{-at(p\|x\|_p^{-1})^\alpha}$;
- 2) $Z(x, t) = \sum_{m=1}^{\infty} \frac{(-1)^m}{m!} \frac{1-p^{\alpha m}}{1-p^{-\alpha m-n}} (at)^m \|x\|_p^{-\alpha m-n}$ for $x \neq 0$;
- 3) $Z(x, t) \geq 0$, for all $x \in \mathbb{Q}_p^n$, $t \in (0, T]$.

Proof. 1) By expanding $Z(x, t)$ as

$$Z(x, t) = \sum_{k=-\infty}^{\infty} \int_{\|\xi\|_p=p^k} \Psi(x \cdot \xi) e^{-at\|\xi\|_p^\alpha} d^n \xi,$$

and applying

$$\int_{\|\xi\|_p=p^k} \Psi(x \cdot \xi) d^n \xi = \begin{cases} p^{kn}(1 - p^{-n}), & \text{if } \|x\|_p \leq p^{-k} \\ -p^{kn}p^{-n}, & \text{if } \|x\|_p = p^{-k+1} \\ 0, & \text{if } \|x\|_p > p^{-k+1}, \end{cases}$$

(c.f. Lemma 4.1 in [25, Chap. III]), we obtain

$$Z(x, t) = (1 - p^{-n})\|x\|_p^{-n} \sum_{k=0}^{\infty} p^{-kn} e^{-at(p^{-k}\|x\|_p^{-1})^\alpha} - \|x\|_p^{-n} e^{-at(p\|x\|_p^{-1})^\alpha}. \quad (3.3)$$

Note that by the previous expansion $Z(x, t)$ is a real-valued function.

2) By using the Taylor expansion of e^x in (3.3), and exchanging the order of summation, and sum the geometric progression, we find that

$$Z(x, t) = \sum_{m=1}^{\infty} \frac{(-1)^m}{m!} \frac{1 - p^{\alpha m}}{1 - p^{-\alpha m-n}} (at)^m \|x\|_p^{-\alpha m-n}, \text{ for } x \neq 0.$$

3) Let $\Omega_l(x)$ denote the characteristic function of the ball $B_{-l}^n(0)$. Then $\mathcal{F}\Omega_l = p^{-nl}\Omega_{-l}$. The last part follows from this observation by means of the following calculation:

$$\begin{aligned} Z(x, t) &= \sum_{l=-\infty}^{\infty} e^{-atp^{l\alpha}} \int_{\|\xi\|_p=p^l} \Psi(x \cdot \xi) d^n \xi \\ &= \sum_{l=-\infty}^{\infty} e^{-atp^{l\alpha}} (p^{n(l)}\Omega_{-l}(x) - p^{n(l-1)}\Omega_{-l+1}(x)) \\ &= \sum_{l=-\infty}^{\infty} p^{nl} (e^{-atp^{l\alpha}} - e^{-atp^{(l+1)\alpha}})\Omega_{-l}(x) \geq 0 \end{aligned}$$

□

Lemma 3.

$$Z(x, t) \leq Ct(t^{1/\alpha} + \|x\|_K)^{-\alpha-n}, \quad t > 0, \quad x \in \mathbb{Q}_p^n. \quad (3.4)$$

Proof. Let l an integer such that $p^{l-1} \leq t^{1/\alpha} \leq p^l$. Then

$$\begin{aligned} Z(x, t) &\leq \int_{\mathbb{Q}_p^n} e^{-at\|\xi\|_p^\alpha} d^n \xi \leq \int_{\mathbb{Q}_p^n} e^{-ap^{\alpha(l-1)}\|\xi\|_p^\alpha} d^n \xi = \int_{\mathbb{Q}_p^n} e^{-a\|p^{-(l-1)}\xi\|_p^\alpha} d^n \xi \\ &= p^{-(l-1)n} \int_{\mathbb{Q}_p^n} e^{-a\|\eta\|_p^\alpha} d\eta = C_0(\alpha) p^{-n} p^{-ln} \leq C_1 t^{-n/\alpha}. \end{aligned} \quad (3.5)$$

On the other hand, if $\|x\|_p \geq t^{1/\alpha}$, by applying Lemma 2 (2), we have

$$Z(x, t) \leq \|x\|_p^{-n} \sum_{m=1}^{\infty} \frac{C_2^m}{m!} (t\|x\|_p^{-\alpha})^m \leq C_3 t \|x\|_p^{-\alpha-n}. \quad (3.6)$$

The result follows from (3.5-3.6) as follows. If $\|x\|_p \geq t^{1/\alpha}$, by (3.6),

$$Z(x, t) \leq C_3 t \|x\|_p^{-\alpha-n} \leq 2^{\alpha+n} C_3 t (t^{1/\alpha} + \|x\|_p)^{-\alpha-n}.$$

If $\|x\|_p < t^{1/\alpha}$, by (3.5),

$$Z(x, t) \leq C_1 t^{-n/\alpha} \leq 2^{\alpha+n} C_1 t (t^{1/\alpha} + \|x\|_p)^{-\alpha-n}.$$

□

Inequality (3.4) shows in particular that the function $Z(x, t)$ belongs, with respect to x , to $L_1(\mathbb{Q}_p^n) \cap L_2(\mathbb{Q}_p^n)$.

Corollary 1.

$$\int_{\mathbb{Q}_p^n} Z(x, t) d^n x = 1. \quad (3.7)$$

3.2. The Spaces \mathfrak{M}_λ and Pseudo-differentiability of the Fundamental Solution.

Definition 2. Denote by $\mathfrak{M}_\lambda, \lambda > 0$, the set of the complex-valued locally constant functions $\varphi(x)$ on \mathbb{Q}_p^n such that

$$|\varphi(x)| \leq C(\varphi) (1 + \|x\|_p^\lambda).$$

If the function φ depends also on a parameter t , we shall say that $\varphi \in \mathfrak{M}_\lambda$ uniformly with respect to t , if its constant C and its exponent of local constancy $l(\varphi)$ do not depend on t .

Lemma 4. If $\varphi \in \mathfrak{M}_\lambda, \lambda < \alpha$, with α as in (3.1), then

$$\lim_{t \rightarrow 0^+} \int_{\mathbb{Q}_p^n} Z(x - \xi, t) \varphi(\xi) d^n \xi = \varphi(x). \quad (3.8)$$

Proof. By Corollary (1) and Lemmas 2 (part 3) and 3 we have

$$\begin{aligned} \left| \int_{\mathbb{Q}_p^n} Z(x - \xi, t) \varphi(\xi) d^n \xi - \varphi(x) \right| &= \left| \int_{\mathbb{Q}_p^n} Z(x - \xi, t) (\varphi(\xi) - \varphi(x)) d^n \xi \right| \\ &\leq \int_{\mathbb{Q}_p^n} Z(x - \xi, t) |\varphi(\xi) - \varphi(x)| d^n \xi \\ &\leq C \int_{\mathbb{Q}_p^n} t(t^{1/\alpha} + \|x - \xi\|_p)^{-\alpha-n} |\varphi(\xi) - \varphi(x)| d^n \xi := I(x, t) \end{aligned}$$

Let η be the exponent of locally constancy of φ . Since $\varphi \in \mathfrak{M}_\lambda$, $\lambda < \alpha$, we can re-write $I(x, t)$ as follows:

$$\begin{aligned} I(x, t) &= C \int_{\|\xi - x\|_p > p^n} t(t^{1/\alpha} + \|\xi - x\|_p)^{-\alpha-n} |\varphi(\xi) - \varphi(x)| d^n \xi \\ &\leq I_1(x, t) + I_2(x, t), \end{aligned}$$

with

$$\begin{aligned} I_1(x, t) &:= C_1 t \int_{\|\xi - x\|_p > p^n} \frac{1 + \|\xi\|_p^\lambda}{(t^{1/\alpha} + \|\xi - x\|_p)^{\alpha+n}} d^n \xi, \\ I_2(x, t) &:= Ct|\varphi(x)| \int_{\|\xi - x\|_p > p^n} (t^{1/\alpha} + \|\xi - x\|_p)^{-\alpha-n} d^n \xi. \end{aligned}$$

Now, since $\alpha > 0$, and $t > 0$,

$$I_2(x, t) \leq C_2 t |\varphi(x)|,$$

and since $\lambda < \alpha$,

$$\begin{aligned} I_1(x, t) &\leq C_1 t \left(C_3 + \int_{\|\tau\|_p > p^n} \frac{\|x - \tau\|_p^\lambda}{\|\tau\|_p^{\alpha+n}} d^n \xi \right) \leq \\ &C_1 t \left(C_3 + \int_{p^n < \|\tau\|_p \leq \|x\|_p} \frac{\|x - \tau\|_p^\lambda}{\|\tau\|_p^{\alpha+n}} d^n \xi + \int_{\|\tau\|_p > \|x\|_p} \frac{\|x - \tau\|_p^\lambda}{\|\tau\|_p^{\alpha+n}} d^n \xi \right) = \\ &C_1 t \left(C_4(x) + \int_{\|\tau\|_p > \|x\|_p} \frac{1}{\|\tau\|_p^{\alpha-\lambda+n}} d^n \xi \right) = C_5(x) t. \end{aligned}$$

Therefore

$$\lim_{t \rightarrow 0^+} \left| \int_{\mathbb{Q}_p^n} Z(x - \xi, t) \varphi(\xi) d^n \xi - \varphi(x) \right| \leq \lim_{t \rightarrow 0^+} C_6(x) t = 0.$$

□

Proposition 1. *The fundamental solution has the following properties:*

- (1) $Z(x, t) \geq 0$, for all $x \in \mathbb{Q}_p^n$, $t \in (0, T]$.
- (2) $\int_{\mathbb{Q}_p^n} Z(x, t) d^n x = 1$, for any $t > 0$;
- (3) if $\varphi \in S(\mathbb{Q}_p^n)$, then $\lim_{(x,t) \rightarrow (x_0,0)} \int_{\mathbb{Q}_p^n} Z(x - \eta, t) \varphi(\eta) d^n \eta = \varphi(x_0)$;
- (4) $Z(x, t + t') = \int_{\mathbb{Q}_p^n} Z(x - y, t) Z(y, t') d^n y$, for $t, t' > 0$.

Proof. (1), (2), and (3) are already established (c.f. Lemma 2-part (3), Corollary 1, and Lemma 4). The last assertion is proved as follows: since $e^{-at\|\xi\|_p^\alpha} \in L^1(\mathbb{Q}_p^n)$,

$$\begin{aligned} \int_{\mathbb{Q}_p^n} Z(x - y, t_1) Z(y, t_2) d^n y &= \mathcal{F}^{-1}(\mathcal{F}(Z(y, t_1) * Z(y, t_2))) \\ &= \mathcal{F}^{-1}\left(e^{-at_1\|\xi\|_p^\alpha} e^{-at_2\|\xi\|_p^\alpha}\right) \\ &= Z(x, t_1 + t_2). \end{aligned}$$

□

Proposition 2. *If $b > 0$, $0 \leq \lambda < \alpha$, and $x \in \mathbb{Q}_p^n$, then*

$$I(b, x) = \int_{\mathbb{Q}_p^n} (b + \|x - \xi\|_p)^{-\alpha-n} \|\xi\|_p^\lambda d^n \xi \leq Cb^{-\alpha} (1 + \|x\|_p^\lambda),$$

where the constant C does not depend on b, x .

Proof. Let m be an integer such that $p^{m-1} \leq b \leq p^m$. Then

$$(b + \|x - \xi\|_p)^{-\alpha-n} \leq (p^{m-1} + \|x - \xi\|_p)^{-\alpha-n},$$

and

$$\begin{aligned} I(b, x) &\leq I(p^{m-1}, x) = \int_{\mathbb{Q}_p^n} (p^{m-1} + \|x - \xi\|_p)^{-\alpha-n} \|\xi\|_p^\lambda d^n \xi \\ &= p^{(m-1)(-\alpha-n)} \int_{\mathbb{Q}_p^n} (1 + \|p^{m-1}x - p^{m-1}\xi\|_p)^{-\alpha-n} \|\xi\|_p^\lambda d^n \xi \\ &= p^{(m-1)(\lambda-\alpha)} \int_{\mathbb{Q}_p^n} (1 + \|p^{m-1}x - \eta\|_p)^{-\alpha-n} \|\eta\|_p^\lambda d^n \eta \\ &= p^{(m-1)(\lambda-\alpha)} I(1, p^{m-1}x). \end{aligned} \tag{3.9}$$

Let $p^{m-1}x = y$, $\|y\|_p = p^l$. We have

$$I(1, y) = I_1(y) + I_2(y) + I_3(y),$$

where

$$\begin{aligned} I_1(y) &= \sum_{k=-\infty}^{l-1} \int_{\|\eta\|_p=p^k} (1 + \|y - \eta\|_p)^{-\alpha-n} \|\eta\|_p^\lambda d^n \eta, \\ I_2(y) &= \int_{\|\eta\|_p=p^l} (1 + \|y - \eta\|_p)^{-\alpha-n} \|\eta\|_p^\lambda d^n \eta, \\ I_3(y) &= \sum_{k=l+1}^{\infty} \int_{\|\eta\|_p=p^k} (1 + \|y - \eta\|_p)^{-\alpha-n} \|\eta\|_p^\lambda d^n \eta. \end{aligned}$$

The results follows from the following estimations:

Claim A. $I_1(y) \leq C_0(1 + \|y\|_p)^{-\alpha-n} \|y\|_p^{\lambda+n}$;

Claim B. $I_2(y) \leq C_1 \|y\|_p^\lambda$;

Claim C. $I_3(y) \leq C_2$.

Indeed, from the claims we have $I(1, y) \leq C_3(1 + \|y\|_p^\lambda)$, and by (3.9),

$$\begin{aligned} I(b, x) &\leq C_3 p^{(m-1)(\lambda-\alpha)} (1 + p^{(1-m)\lambda} \|x\|_p^\lambda) \\ &\leq C_3 p^{-m\alpha} (1 + \|x\|_p^\lambda) \leq Cb^{-\alpha} (1 + \|x\|_p^\lambda). \end{aligned}$$

We now prove the announced claims.

Proof of Claim A.

$$\begin{aligned} I_1(y) &= \sum_{k=-\infty}^{l-1} \int_{\|\eta\|_p=p^k} (1 + \|y - \eta\|_p)^{-\alpha-n} \|\eta\|_p^\lambda d^n \eta, \\ &= (1 - p^{-n})(1 + \|y\|_p)^{-\alpha-n} \sum_{k=-\infty}^{l-1} p^{(\lambda+n)k} \\ &\leq C_0(1 + \|y\|_p)^{-\alpha-n} \|y\|_p^{\lambda+n}, \end{aligned}$$

where

$$C_0 = \frac{(1 - p^{-n})p^{-\lambda-n}}{1 - p^{-\lambda-n}}.$$

Proof of Claim B. Let $\tilde{y} \in \mathbb{Q}_p$ such that $|\tilde{y}|_p = p^l = \|y\|_p$, then

$$\begin{aligned} I_2(y) &= \int_{\|\eta\|_p=p^l} (1 + \|y - \eta\|_p)^{-\alpha-n} \|\eta\|_p^\lambda d^n \eta \\ &= \|y\|_p^\lambda \int_{\|\eta\|_p=p^l} (1 + |\tilde{y}|_p |\tilde{y}^{-1}y - \tilde{y}^{-1}\eta|_p)^{-\alpha-n} d^n \eta \\ &= \|y\|_p^{\lambda-\alpha} \int_{\|\eta\|_p=1} (\|y\|_p^{-1} + \|u - \eta\|_p)^{-\alpha-n} d^n \eta, \text{ with } u = \tilde{y}^{-1}y. \end{aligned}$$

We set

$$A_m = \{\eta \in \mathbb{Q}_p^n \mid \|\eta\|_p = 1 \text{ and } \|u - \eta\|_p = p^{-m}\}, \text{ for } m \in \mathbb{N},$$

and for I non-empty subset of $\{1, 2, \dots, n\}$,

$$A_{m,I} = \{\eta \in A_m \mid |u_i - \eta_i|_p = p^{-m} \text{ for } i \in I \text{ and } |u_i - \eta_i|_p < p^{-m} \text{ for } i \notin I\},$$

where $u = (u_1, \dots, u_n)$, $\eta = (\eta_1, \dots, \eta_n) \in \mathbb{Q}_p^n$, with $\|\eta\|_p = \|u\|_p = 1$.

With this notation we have $A_m \subseteq \bigcup_I A_{m,I}$,

$$\text{vol}(A_{m,I}) \leq (p^{-m}(1 - p^{-1}))^{|I|} (p^{-m-1})^{n-|I|},$$

here $|I|$ denotes the cardinality of I , then

$$\text{vol}(A_m) \leq \sum_{|I|=0}^n \binom{n}{|I|} (p^{-m}(1 - p^{-1}))^{|I|} (p^{-m-1})^{n-|I|} = p^{-mn},$$

and

$$\begin{aligned} I_2(y) &= \|y\|_p^{\lambda-\alpha} \sum_{m=0}^{\infty} \int_{A_m} (\|y\|_p^{-1} + \|u - \eta\|_p)^{-\alpha-n} d^n \eta \\ &\leq \|y\|_p^{\lambda-\alpha} \sum_{m=0}^{\infty} (\|y\|_p^{-1} + p^{-m})^{-\alpha-n} p^{-mn} \\ &= \frac{\|y\|_p^{\lambda-\alpha}}{1 - p^{-n}} \sum_{m=0}^{\infty} \int_{\|\eta\|_p=p^{-m}} (\|y\|_p^{-1} + \|\eta\|_p)^{-\alpha-n} d^n \eta \\ &= \frac{\|y\|_p^{\lambda-\alpha}}{1 - p^{-n}} \int_{\|\eta\|_p \leq 1} (\|y\|_p^{-1} + \|\eta\|_p)^{-\alpha-n} d^n \eta \\ &\leq C'_1 \|y\|_p^{\lambda-\alpha} \int_{\mathbb{Q}_p^n} (\|y\|_p^{-1} + \|\eta\|_p)^{-\alpha-n} d^n \eta \\ &= C'_1 \|y\|_p^{\lambda-\alpha} \int_{\mathbb{Q}_p^n} (\|y\|_p^{-1} + \|y\|_p^{-1} |\tilde{y}\eta|_p)^{-\alpha-n} d^n \eta \\ &= C'_1 \|y\|_p^{\lambda+n} \int_{\mathbb{Q}_p^n} (1 + \|\tilde{y}\eta\|_p)^{-\alpha-n} d^n \eta \\ &= C'_1 \|y\|_p^\lambda \int_{\mathbb{Q}_p^n} (1 + \|\tau\|_p)^{-\alpha-n} d\tau \leq C_1 \|y\|_p^\lambda. \end{aligned}$$

Proof of Claim C.

$$\begin{aligned} I_3(y) &= \sum_{k=l+1}^{\infty} \int_{\|\eta\|_p=p^k} (1 + \|\eta\|_p)^{-\alpha-n} \|\eta\|_p^\lambda d^n \eta, \\ &\leq \int_{\mathbb{Q}_p^n} (1 + \|\eta\|_p)^{-\alpha-n} \|\eta\|_p^\lambda d^n \eta = C. \end{aligned}$$

□

Lemma 5. *If $\alpha > 0$, then*

$$\|x\|_p^\alpha = \frac{1}{\Gamma_p^{(n)}(-\alpha)} \int_{\mathbb{Q}_p^n} \|y\|_p^{-\alpha-n} (\Psi(-x \cdot y) - 1) d^n y \quad (3.10)$$

for all $x \in \mathbb{Q}_p^n$.

Proof. The proof is a slightly variation of the proof of Proposition 2.3 in [18]. □

Lemma 6. *Let $0 < \gamma \leq \alpha$, then*

$$(D_T^\gamma Z)(x, t) = \int_{\mathbb{Q}_p^n} \Psi(x \cdot \eta) \|\eta\|_p^\gamma e^{-at\|\eta\|_p^\alpha} d^n \eta.$$

Proof. By Lemma 2 (2), $Z(x - y, t) = Z(x, t)$, for $\|y\| < \|x\|$. Then we can use (2.3) to calculate $(D_T^\gamma Z)(x, t)$:

$$\begin{aligned} (D_T^\gamma Z)(x, t) &= \frac{1}{\Gamma_p^{(n)}(-\gamma)} \int_{\mathbb{Q}_p^n} \|y\|_p^{-\gamma-n} (Z(x - y, t) - Z(x, t)) d^n y \\ &= \frac{1}{\Gamma_p^{(n)}(-\gamma)} \int_{\|y\|_p \geq \|x\|_p} \|y\|_p^{-\gamma-n} (Z(x - y, t) - Z(x, t)) d^n y. \end{aligned}$$

We now use Lemma 3 to obtain

$$\begin{aligned} |(D_T^\gamma Z)(x, t)| &\leq \left| \frac{1}{\Gamma_p^{(n)}(-\gamma)} \right| \int_{\|y\|_p \geq \|x\|_p} (Ct\|y\|_p^{-\gamma-\alpha-2n} + Z(x, t)\|y\|_p^{-\gamma-n}) d^n y \\ &< \infty. \end{aligned} \quad (3.11)$$

This shows that $(D_T^\gamma Z)(x, t)$ exists. We now compute this function explicitly.

We set

$$Z^{(m)}(x, t) := \int_{\|\xi\|_p \leq p^m} \Psi(x \cdot \xi) e^{-at\|\xi\|_p^\alpha} d^n \xi.$$

Then $Z^{(m)}(x, t)$ is bounded and locally constant as function of x , the exponent of local constancy is m . From these observations by using Lemma 2 (2), and (2.3) we calculate $(D_T^\gamma Z)(x, t)$ as follows:

$$\begin{aligned} (D_T^\gamma Z^{(m)})(x, t) &= \frac{1}{\Gamma_p^{(n)}(-\gamma)} \int_{\mathbb{Q}_p^n} \|y\|_p^{-\gamma-n} (Z^{(m)}(x - y, t) - Z^{(m)}(x, t)) d^n y \\ &= \frac{1}{\Gamma_p^{(n)}(-\gamma)} \int_{\|y\|_p > p^{-m}} \|y\|_p^{-\gamma-n} (Z^{(m)}(x - y, t) - Z^{(m)}(x, t)) d^n y \\ &= \frac{1}{\Gamma_p^{(n)}(-\gamma)} \int_{\|y\|_p > p^{-m}} \|y\|_p^{-\gamma-n} \int_{\|\eta\|_p \leq p^m} e^{-at\|\eta\|_p^\alpha} \Psi(x \cdot \eta) (\Psi(-y \cdot \eta) - 1) d^n \eta d^n y \end{aligned}$$

$$= \int_{\|\eta\|_p \leq p^m} e^{-at\|\eta\|_p^\alpha} \Psi(x \cdot \eta) \times \left(\frac{1}{\Gamma_p^{(n)}(-\gamma)} \int_{\|y\|_p > p^{-m}} \|y\|_p^{-\gamma-n} (\Psi(-y \cdot \eta) - 1) d^n y \right) d^n \eta.$$

Note that if $\|y\|_p \leq p^{-m}$, then $\Psi(-y \cdot \eta) = 1$ for all η such that $\|\eta\|_p \leq p^m$, using this observation and Lemma (5), $(D_T^\gamma Z^{(m)})(x, t)$ becomes

$$\begin{aligned} & \int_{\|\eta\|_p \leq p^m} e^{-at\|\eta\|_p^\alpha} \Psi(x \cdot \eta) \left(\frac{1}{\Gamma_p^{(n)}(-\gamma)} \int_{\mathbb{Q}_p^n} \|y\|_p^{-\gamma-n} (\Psi(-y \cdot \eta) - 1) d^n y \right) d^n \eta \\ &= \int_{\|\eta\|_p \leq p^m} e^{-at\|\eta\|_p^\alpha} \Psi(x \cdot \eta) \|\eta\|_p^\gamma d^n \eta. \end{aligned}$$

By the dominated convergence theorem and (3.11) we have

$$(D_T^\gamma Z)(x, t) = \int_{\mathbb{Q}_p^n} e^{-at\|\eta\|_p^\alpha} \Psi(x \cdot \eta) \|\eta\|_p^\gamma d^n \eta.$$

□

Lemma 7.

$$\begin{aligned} \frac{\partial Z}{\partial t}(x, t) &= -a \int_{\mathbb{Q}_p^n} \Psi(x \cdot \xi) \|\xi\|_p^\alpha e^{-at\|\xi\|_p^\alpha} d^n \xi; \\ \frac{\partial Z}{\partial t}(x, t) &= -a(D_T^\gamma Z)(x, t), \text{ for } 0 < \gamma \leq \alpha. \end{aligned}$$

Proof. The first part follows by applying the dominated convergence theorem. The second part follows from the first one by Lemma 6. □

Lemma 8.

$$\begin{aligned} \left| \frac{\partial Z}{\partial t}(x, t) \right| &\leq C \left(t^{1/\alpha} + \|x\|_p \right)^{-\alpha-n}; \\ |(D_T^\gamma Z)(x, t)| &\leq C \left(t^{1/\alpha} + \|x\|_p \right)^{-\gamma-n}. \end{aligned}$$

Proof. The proof uses the same reasonings as the one given in the proof of Lemma 3. □

Corollary 2.

$$\int_{\mathbb{Q}_p^n} (D_T^\gamma Z)(x, t) d^n x = 0$$

3.3. The Cauchy Problem for the multidimensional p -adic Heat Equation.

Theorem 1. Let $\varphi(x)$, $f(x, t) \in \mathfrak{M}_\lambda$, $0 \leq \lambda < \alpha$ be continuous functions. Then the Cauchy problem

$$\begin{cases} \frac{\partial u(x, t)}{\partial t} + a(D_T^\alpha u)(x, t) = f(x, t), & x \in \mathbb{Q}_p^n, \quad t \in (0, T], \\ u(x, 0) = \varphi(x), \end{cases} \quad (3.12)$$

with $a > 0$, $\alpha > 0$, has a continuous solution in \mathfrak{M}_λ given by

$$u(x, t) = \int_{\mathbb{Q}_p^n} Z(x - \xi, t) \varphi(\xi) d^n \xi + \int_0^t \left(\int_{\mathbb{Q}_p^n} Z(x - \xi, t - \tau) f(\xi, \tau) d^n \xi \right) d\tau.$$

The proof of the theorem will be accomplished through the following lemmas.
We set

$$u_1(x, t) := \int_{\mathbb{Q}_p^n} Z(x - \xi, t) \varphi(\xi) d^n \xi, \quad \text{and}$$

$$u_2(x, t) := \int_0^t \left(\int_{\mathbb{Q}_p^n} Z(x - \xi, t - \tau) f(\xi, \tau) d^n \xi \right) d\tau.$$

Lemma 9. $u(x, t) \in \mathfrak{M}_\lambda$ uniformly with respect to t , and $u(x, t)$ satisfies the initial conditions of Theorem 1.

Proof. We first show that $u_1(x, t) \in \mathfrak{M}_\lambda$ uniformly with respect to t . Since φ is locally constant, there exist $l \in \mathbb{N}$ such that $\varphi(\xi + y) = \varphi(\xi)$ for any $\|y\|_p \leq p^{-l}$. By changing variables $y - \xi = -\eta$ in $u_1(x, t)$ we that $u_1(x, t)$ is locally constant. Now using Lemma 3 and Proposition 2 we have $|u_1(x, t)| \leq C(1 + \|x\|)^\lambda$, and thus $u_1(x, t) \in \mathfrak{M}_\lambda$ uniformly with respect to t .

By a similar reasoning one shows that $u_2(x, t)$ is locally constant in x , and that $|u_2(x, t)| \leq CT(1 + \|x\|)^\lambda$. Therefore $u(x, t) = u_1(x, t) + u_2(x, t) \in \mathfrak{M}_\lambda$ uniformly with respect to t .

We now show that $\lim_{t \rightarrow 0^+} u(x, t) = \varphi(x)$: by Lemma 4, $\lim_{t \rightarrow 0^+} u_1(x, t) = \varphi(x)$, and $\lim_{t \rightarrow 0^+} u_2(x, t) = 0$, since $|u_2(x, t)| \leq Ct(1 + \|x\|)^\lambda$, $t \leq T$. \square

We now compute the partial derivatives of $u_1(x, t)$, $u_2(x, t)$ with respect to t .

Lemma 10.

$$\frac{\partial u_1}{\partial t}(x, t) = \int_{\mathbb{Q}_p^n} \frac{\partial Z}{\partial t}(x - \xi, t) \varphi(\xi) d^n \xi.$$

Proof. The results follows by applying the dominated convergence theorem. \square

Lemma 11.

$$\frac{\partial u_2}{\partial t}(x, t) = \int_0^t \left(\int_{\mathbb{Q}_p^n} \frac{\partial Z}{\partial t}(x - \xi, t - \tau) (f(\xi, \tau) - f(x, \tau)) d^n \xi \right) d\tau + f(x, t).$$

Proof. Let

$$u_{2,h}(x, t) := \int_0^{t-h} \left(\int_{\mathbb{Q}_p^n} Z(x - \xi, t - \tau) f(\xi, \tau) d^n \xi \right) d\tau,$$

where h is a small positive number. Then $\frac{u_h(x, t+t') - u_h(x, t)}{t'}$ equals

$$\begin{aligned} & \int_0^{t-h} \left(\int_{\mathbb{Q}_p^n} \frac{Z(x - \xi, t+t' - \tau) - Z(x - \xi, t - \tau)}{t'} f(\xi, \tau) d^n \xi \right) d\tau \\ & + \int_{t-h}^{t-h+t'} \left(\int_{\mathbb{Q}_p^n} \frac{Z(x - \xi, t+t' - \tau) - Z(x - \xi, t - \tau)}{t'} f(\xi, \tau) d^n \xi \right) d\tau \\ & + \frac{1}{t'} \int_{t-h}^{t-h+t'} \left(\int_{\mathbb{Q}_p^n} Z(x - \xi, t - \tau) f(\xi, \tau) d^n \xi \right) d\tau. \end{aligned} \quad (3.13)$$

By taking $t' \rightarrow 0^+$ the first integral in (??) tends to

$$\int_0^{t-h} \left(\int_{\mathbb{Q}_p^n} \frac{\partial Z}{\partial t}(x - \xi, t - \tau) f(\xi, \tau) d^n \xi \right) d\tau,$$

and by using the continuity of the functions

$$\begin{aligned} & \int_{\mathbb{Q}_p^n} (Z(x - \xi, t + t' - \tau) - Z(x - \xi, t - \tau)) f(\xi, \tau) d^n \xi, \quad \text{and} \\ & \int_{\mathbb{Q}_p^n} Z(x - \xi, t - \tau) f(\xi, \tau) d^n \xi, \end{aligned}$$

with respect to τ , the second integral in (3.13) tends to zero, and the third integral tends to

$$\int_{\mathbb{Q}_p^n} Z(x - \xi, h) f(\xi, t - h) d^n \xi.$$

Hence

$$\begin{aligned} \frac{\partial u_{2,h}}{\partial t}(x, t) &= \int_0^{t-h} \left(\int_{\mathbb{Q}_p^n} \frac{\partial Z}{\partial t}(x - \xi, t - \tau) f(\xi, \tau) d^n \xi \right) d\tau \\ &+ \int_{\mathbb{Q}_p^n} Z(x - \xi, h) f(\xi, t - h) d^n \xi. \end{aligned}$$

This expression can be re-written as

$$\begin{aligned} \frac{\partial u_{2,h}}{\partial t}(x, t) &= \int_0^{t-h} \left(\int_{\mathbb{Q}_p^n} \frac{\partial Z}{\partial t}(x - \xi, t - \tau) (f(\xi, \tau) - f(x, \tau)) d^n \xi \right) d\tau \\ &+ \int_0^{t-h} \left(\int_{\mathbb{Q}_p^n} \frac{\partial Z}{\partial t}(x - \xi, t - \tau) f(x, \tau) d^n \xi \right) d\tau \\ &+ \int_{\mathbb{Q}_p^n} Z(x - \xi, h) (f(\xi, t - h) - f(x, t)) d^n \xi \\ &+ \int_{\mathbb{Q}_p^n} Z(x - \xi, h) f(x, t) d^n \xi. \end{aligned} \tag{3.14}$$

The first integral contains no singularity at $t = \tau$ due to Lemma 8 and the local constancy of f . By Corollary 2, the second integral in (3.14) is equal to zero. The third integral can be written as the sum of the integrals over $\{\xi \in \mathbb{Q}_p^n \mid \|\xi\|_p \leq p^m\}$ and its complement; one integral is estimated on the basis of uniform continuity of f , while the other contains no singularity. Hence this integral tends to zero as h approaches zero from the right. By Lemma 4, the fourth integral tends to $f(x, t)$ as $h \rightarrow 0^+$, therefore

$$\frac{\partial u_2}{\partial t}(x, t) = \int_0^t \left(\int_{\mathbb{Q}_p^n} \frac{\partial Z}{\partial t}(x - \xi, t - \tau) (f(\xi, \tau) - f(x, t)) d^n \xi \right) d\tau + f(x, t).$$

□

As a consequence of Lemmas 10-11 we obtain:

Proposition 3.

$$\begin{aligned} \frac{\partial u}{\partial t}(x, t) &= \int_{\mathbb{Q}_p^n} \frac{\partial Z}{\partial t}(x - \xi, t) \varphi(\xi) d^n \xi \\ &+ \int_0^t \left(\int_{\mathbb{Q}_p^n} \frac{\partial Z}{\partial t}(x - \xi, t - \tau) (f(\xi, \tau) - f(x, t)) d^n \xi \right) d\tau + f(x, t). \end{aligned}$$

We now consider the action of the operator D_T^γ , $0 < \gamma \leq \alpha$ upon $u(x, t)$. We first note that $(D_T^\gamma u)(x, t)$ is defined if $\gamma > \lambda$. This follows from (2.3) using $u(x, t) \in \mathfrak{M}_\lambda$.

Lemma 12. *Let $\lambda < \gamma \leq \alpha$, then*

$$(D_T^\gamma u_1)(x, t) = \int_{\mathbb{Q}_p^n} (D_T^\gamma Z)(x - \xi, t) \varphi(\xi) d^n \xi.$$

Proof. Let $Z_\gamma(x, t) := (D_T^\gamma Z)(x, t)$ and

$$Z_{\gamma, l}(x, t) := \frac{1}{\Gamma_p^{(n)}(-\gamma)} \int_{\|y\|_p > p^{-l}} \|y\|_p^{-\gamma-n} (Z(x-y, t) - Z(x, t)) d^n y. \quad (3.15)$$

By the Fubini theorem

$$\begin{aligned} &\frac{1}{\Gamma_p^{(n)}(-\gamma)} \int_{\|y\|_p > p^{-l}} \|y\|_p^{-\gamma-n} (u_1(x-y, t) - u_1(x, t)) d^n y \\ &= \frac{1}{\Gamma_p^{(n)}(-\gamma)} \int_{\|y\|_p > p^{-l}} \|y\|_p^{-\gamma-n} \left(\int_{\mathbb{Q}_p^n} (Z(x-y-\xi, t) - Z(x-\xi, t)) \varphi(\xi) d^n \xi \right) d^n y \\ &= \int_{\mathbb{Q}_p^n} \varphi(\xi) \left(\frac{1}{\Gamma_p^{(n)}(-\gamma)} \int_{\|y\|_p > p^{-l}} \|y\|_p^{-\gamma-n} (Z(x-y-\xi, t) - Z(x-\xi, t)) d^n y \right) d^n \xi \\ &= \int_{\mathbb{Q}_p^n} Z_{\gamma, l}(x-\xi, t) \varphi(\xi) d^n \xi. \end{aligned}$$

Let m a fixed positive integer, then the last integral can expressed as

$$\int_{\|x-\xi\|_p \geq p^{-m}} Z_{\gamma, l}(x-\xi, t) \varphi(\xi) d^n \xi + \int_{\|x-\xi\|_p < p^{-m}} Z_{\gamma, l}(x-\xi, t) \varphi(\xi) d^n \xi.$$

Now if $\|x\|_p \geq p^{-m}$, $l > m$, then $Z_{\gamma, l}(x, t) = Z_\gamma(x, t)$, and

$$\begin{aligned} &\frac{1}{\Gamma_p^{(n)}(-\gamma)} \int_{\|y\|_p > p^{-l}} \|y\|_p^{-\gamma-n} (u_1(x-y, t) - u_1(x, t)) dy = \\ &\quad \int_{\|x-\xi\|_p \geq p^{-m}} Z_\gamma(x-\xi, t) \varphi(\xi) d\xi \\ &\quad + \int_{\|x-\xi\|_p < p^{-m}} Z_{\gamma, l}(x-\xi, t) \varphi(\xi) d\xi, \end{aligned} \quad (3.16)$$

for $l > m$. Now using Fubini's theorem, and taking $\lim_{l \rightarrow \infty}$, we obtain that

$$\lim_{l \rightarrow \infty} \int_{\|x-\xi\|_p < p^{-m}} Z_{\gamma, l}(x-\xi, t) \varphi(\xi) d\xi$$

$$\begin{aligned}
 &= \int_{\mathbb{Q}_p^n} \|y\|_p^{-\gamma-n} \times \\
 &\quad \left(\frac{1}{\Gamma_p^{(n)}(-\gamma)} \int_{\|x-\xi\|_p < p^{-m}} (Z(x-\xi-y, t) - Z(x-\xi, t)) \varphi(\xi) d^n \xi \right) d^n y \\
 &= \int_{\|x-\xi\|_p < p^{-m}} \times \\
 &\quad \left(\frac{1}{\Gamma_p^{(n)}(-\gamma)} \int_{\mathbb{Q}_p^n} \|y\|_p^{-\gamma-n} (Z(x-\xi-y, t) - Z(x-\xi, t)) d^n y \right) \varphi(\xi) d^n \xi \\
 &= \int_{\|x-\xi\|_p < p^{-m}} Z_\gamma(x-\xi, t) \varphi(\xi) d\xi \tag{3.17}
 \end{aligned}$$

Since

$$\|y\|_p^{-\gamma-n} (u_1(x-y, t) - u_1(x, t))$$

is integrable as function of y (because $u_1(x, t) \in \mathfrak{M}_\lambda$, $\gamma > \lambda$, by Lemma 9), the result follows by taking $\lim_{l \rightarrow \infty}$ in (3.16) and using (3.17), since \square

Lemma 13. *Let $\lambda < \gamma \leq \alpha$, then*

$$(D_T^\gamma u_2)(x, t) = \int_0^t \left(\int_{\mathbb{Q}_p^n} (D_T^\gamma Z)(x-\xi, t-\tau) f(\xi, \tau) d\xi \right) d\tau.$$

Proof. We set

$$u_{2,h}(x, t) := \int_0^{t-h} \left(\int_{\mathbb{Q}_p^n} Z(x-y, t-\theta) f(y, \theta) d^n y \right) d\theta.$$

Then

$$\begin{aligned}
 &\frac{1}{\Gamma_p^{(n)}(-\gamma)} \int_{\|y\|_p > p^{-l}} \|y\|_p^{-\gamma-n} (u_{2,h}(x-y, t) - u_{2,h}(x, t)) d^n y \\
 &= \frac{1}{\Gamma_p^{(n)}(-\gamma)} \int_{\|y\|_p > p^{-l}} \|y\|_p^{-\gamma-n} \times \\
 &\quad \left(\int_0^{t-h} \left(\int_{\mathbb{Q}_p^n} (Z(x-y-\xi, t-\tau) - Z(x-\xi, t-\tau)) f(\xi, \tau) d^n \xi \right) d\tau \right) d^n y \\
 &= \int_0^{t-h} \left(\int_{\mathbb{Q}_p^n} Z_{\gamma,l}(x-\xi, t-\tau) f(\xi, \tau) d^n \xi \right) d\tau,
 \end{aligned}$$

with $Z_{\gamma,l}(x, t)$ as in (3.15). We now note that

$$Z_{\gamma,l}(x, t) = \int_{\mathbb{Q}_p^n} \psi(x \cdot \xi) P_l(\xi) e^{-at\|\xi\|_p^\alpha} d\xi,$$

where

$$P_l(\xi) = \frac{1}{\Gamma_p^{(n)}(-\gamma)} \int_{\|y\|_p > p^{-l}} \|y\|_p^{-\gamma-n} (\psi(-y \cdot \xi) - 1) dy.$$

By using a similar reasoning to the one used in [18, pg. 142], we have

$$|P_l(\xi)| \leq \frac{2\|\xi\|_p^\gamma}{|\Gamma_p^{(n)}(-\gamma)|} \int_{\|u\|_p > 1} \|u\|_p^{-\gamma-n} d^n u = C\|\xi\|_p^\gamma,$$

whence

$$|Z_{\gamma,l}(x, t)| \leq C'.$$

Furthermore, if $\|x - \xi\|_p \geq p^{-(l-1)}$ then $Z_{\gamma,l}(x - \xi, t - \tau) = Z_\gamma(x - \xi, t - \tau)$.

Therefore

$$\begin{aligned} & \int_0^{t-h} \left(\int_{\mathbb{Q}_p^n} Z_{\gamma,l}(x - \xi, t - \tau) f(\xi, \tau) d^n \xi \right) d\tau \\ &= \int_0^{t-h} \left(\int_{\|x - \xi\|_p \geq p^{-(l-1)}} Z_\gamma(x - \xi, t - \tau) f(\xi, \tau) d^n \xi \right) d\tau \\ &+ \int_0^{t-h} \left(\int_{\|x - \xi\|_p < p^{-(l-1)}} Z_{\gamma,l}(x - \xi, t - \tau) f(\xi, \tau) d^n \xi \right) d\tau. \end{aligned}$$

By taking $l \rightarrow \infty$ we obtain that

$$\begin{aligned} (D_T^\gamma u_{2,h})(x, t) &= \int_0^{t-h} \left(\int_{\mathbb{Q}_p^n} (D_T^\gamma Z)(x - \xi, t - \tau) f(\xi, \tau) d^n \xi \right) d\tau \\ &= \int_0^{t-h} \left(\int_{\mathbb{Q}_p^n} (D_T^\gamma Z)(x - \xi, t - \tau) (f(\xi, \tau) - f(x, \tau)) d^n \xi \right) d\tau \\ &= \int_0^{t-h} \left(\int_{\|x - \xi\|_p > p^{-l}} (D_T^\gamma Z)(x - \xi, t - \tau) (f(\xi, \tau) - f(x, \tau)) d^n \xi \right) d\tau, \end{aligned}$$

where l is the exponent of local constancy of $f(\xi, \tau)$ (c.f. Corollary 2). Finally, since $u_{2,h} \in \mathcal{M}_\lambda$ uniformly in h (c.f. Lemma 3), by taking $h \rightarrow 0^+$ and using the dominated convergence theorem, we have

$$(D_T^\gamma u_2)(x, t) = \int_0^t \left(\int_{\mathbb{Q}_p^n} (D_T^\gamma Z)(x - \xi, t - \tau) (f(\xi, \tau) - f(x, \tau)) d^n \xi \right) d\tau.$$

□

As a consequence of Lemmas 7, 12, and 13, we obtain the following result.

Proposition 4.

$$\begin{aligned} (D_T^\gamma u)(x, t) &= \int_{\mathbb{Q}_p^n} (D_T^\gamma Z)(x - \xi, t) \varphi(\xi) d^n \xi \\ &+ \int_0^t \left(\int_{\mathbb{Q}_p^n} (D_T^\gamma Z)(x - \xi, t - \tau) (f(\xi, \tau) - f(x, \tau)) d^n \xi \right) d\tau; \end{aligned}$$

$$a(D_T^\gamma u)(x, t) = - \int_{\mathbb{Q}_p^n} \frac{\partial Z}{\partial t}(x - \xi, t) \varphi(\xi) d^n \xi - \int_0^t \left(\int_{\mathbb{Q}_p^n} \frac{\partial Z}{\partial t}(x - \xi, t - \tau) (f(\xi, \tau) - f(x, \tau)) d^n \xi \right) d\tau,$$

for $0 < \gamma \leq \alpha$.

3.3.1. *Proof of Theorem 1.* By Lemma 9, $u(x, t) \in \mathfrak{M}_\lambda$ uniformly with respect to t , and $u(x, t)$ satisfies the initial condition of Theorem 1. By Propositions 3-4, $u(x, t)$ is a solution of Cauchy problem (3.12).

3.4. Taibleson Operator and Elliptic Pseudo-differential Operators. For a polynomial $g(x) \in \mathbb{Z}_p[x_1, \dots, x_n]$ we denote by $\bar{g}(x) \in \mathbb{F}_p[x_1, \dots, x_n]$ its reduction modulo p , i.e., the polynomial obtained by reducing the coefficients of $g(x)$ modulo p . Let $f(x) \in \mathbb{Z}_p[x_1, \dots, x_n]$, $f(0) = 0$, be a non-constant homogeneous polynomial of degree d such that $\bar{f}(x) \neq 0$. We say that $f(x)$ is elliptic modulo p if

$$\{x \in \mathbb{F}_p^n \mid \bar{f}(x) = 0\} = \{0\},$$

and that $f(x)$ is elliptic over \mathbb{Q}_p if

$$\{x \in \mathbb{Q}_p^n \mid f(x) = 0\} = \{0\}.$$

Note that if f elliptic modulo p , then f is elliptic over \mathbb{Q}_p .

If I is a non-empty subset of $\{1, \dots, n\}$, we define $f_I(x)$, respectively $\bar{f}_I(x)$, as the polynomial mapping obtained by restricting $f(x)$ to the set

$$T_I := \{x \in \mathbb{Z}_p^n \mid x_i \neq 0 \Leftrightarrow i \in I\},$$

respectively, to the set

$$\bar{T}_I := \{x \in \mathbb{F}_p^n \mid x_i \neq 0 \Leftrightarrow i \in I\}.$$

Definition 3. Let $f(x) \in \mathbb{Z}_p[x_1, \dots, x_n]$, $f(0) = 0$, be a non-constant homogeneous polynomial of degree d with coefficients in \mathbb{Z}_p^\times . We say that $f(x)$ is strongly elliptic modulo p , if for every non-empty subset I of $\{1, \dots, n\}$, $\bar{f}_I(x)$ is elliptic modulo p .

Example 1. Let $f(x) = x^2 - vy^2$, with $v \in \mathbb{Z}_p^\times \setminus (\mathbb{Z}_p^\times)^2$, where

$$(\mathbb{Z}_p^\times)^2 := \{x \in \mathbb{Z}_p^\times \mid x = y^2, \text{ for some } y \in \mathbb{Z}_p^\times\}.$$

Then $f(x)$ is strongly elliptic modulo p .

Lemma 14. There are infinitely many strongly elliptic polynomials modulo p .

Proof. By induction on n , the number of variables. The case $n = 1$ is clear. Assume as induction hypothesis that the result is true for $1 \leq n \leq k$, $k \geq 2$. Let $g(x_1, \dots, x_k)$ be a strongly elliptic polynomial modulo p of degree d . Set any $v \in \mathbb{Z}_p^\times$ such that \bar{v} does not have a l -th root in \mathbb{F}_p^\times , for some $l \geq 2$, and $f(x_1, \dots, x_{k+1}) = g(x_1, \dots, x_k)^l - vx_{k+1}^l$. Then $f(x_1, \dots, x_{k+1})$ is strongly elliptic modulo p . \square

Lemma 15. Let $f(x) \in \mathbb{Z}_p[x_1, \dots, x_n]$, $f(0) = 0$, be a non-constant homogeneous polynomial of degree d with coefficients in \mathbb{Z}_p^\times . If $f(x)$ is strongly elliptic modulo p , then

$$|f(x)|_p = \|x\|_p^d, \text{ for any } x \in \mathbb{Q}_p^n. \tag{3.18}$$

Proof. We set $A := \{(z_1, \dots, z_n) \in \mathbb{Z}_p^n \mid |z_i|_p = 1, \text{ for some } i\}$. Since $f(x)$ is elliptic over \mathbb{Q}_p ,

$$\left(\sup_{z \in A} |f(z)|_p \right) \|x\|_p^d \leq |f(x)|_p \leq \left(\inf_{z \in A} |f(z)|_p \right) \|x\|_p^d,$$

(c.f. Lemma 1 in [29]). Thus, in order to prove the result it is sufficient to show that

$$|f|_p|_A \equiv 1.$$

Given a non-empty subset I of $\{1, \dots, n\}$, we define

$$A_I = \{x \in A \mid |x_i| = 1 \Leftrightarrow i \in I\}.$$

Then $\cup_I A_I$ is a partition of A when I runs through all non-empty subsets of $\{1, \dots, n\}$, and to show 3.18) it is sufficient to prove that

$$|f|_p|_{A_I} \equiv 1, \text{ for every non-empty subset } I.$$

Without loss of generality we may assume that $I = \{1, \dots, r\}$, $1 \leq r \leq n$. Thus, if $x \in A_I$, then $x_i \in \mathbb{Z}_p^\times$, $i = 1, \dots, r$, and $x_i \in p\mathbb{Z}_p$, $i = r+1, \dots, n$, and $\bar{f}(x) = \bar{f}_I(x) \neq 0$, since f is strongly elliptic modulo p , therefore $|f|_p|_{A_I} \equiv 1$. \square

4. MARKOV PROCESSES AND FUNDAMENTAL SOLUTIONS

Theorem 2. *The fundamental solution $Z(x, t)$ is a transition density of a time- and space-homogeneous non-exploding right continuous strict Markov process without second kind discontinuities.*

Proof. By Proposition 1 (4) the family of operators

$$(\Theta(t)f)(x) = \int_{\mathbb{Q}_p^n} Z(x - \eta, t) f(\eta) d\eta$$

has the semigroup property. We know that $Z(x, t) > 0$ and $\Theta(t)$ preserves the function $f(x) \equiv 1$ (cf. Proposition 1). Thus $\Theta(t)$ is a Markov semigroup. The requiring properties of the corresponding Markov process follow from Proposition 1 and general theorems of the theory of Markov processes [7], see also [28, Section XVI]. \square

REFERENCES

- [1] Albeverio S., and Karwoski W., Diffusion in p -adic numbers. In: K. Ito, H. Hida (Eds.), Gaussian Random Fields, pp. 86-99, 1991, World Scientific, Singapore.
- [2] Albeverio S., and Karwoski W., A random walk on p -adics: the generator and its spectrum, Stochastic Process. Appl. 53 (1994), 1-22.
- [3] Albeverio, S.; Khrennikov, A. Yu.; Shelkovich, V. M., Harmonic analysis in the p -adic Litzorkin spaces: fractional operators, pseudo-differential equations, p -adic wavelets, Tauberian theorems. J. Fourier Anal. Appl. 12 (2006), no. 4, 393-425.
- [4] Avetisov A. V., Bikulov A. H., Kozyrev S. V., and Osipov V. A., p -adic models of ultrametric diffusion constrained by hierarchical energy landscapes, J. Phys. A: Math. Gen. 35 (2002), 177-189.
- [5] Avetisov A. V., Bikulov A. H., and Osipov V. A., p -adic description of characteristic relaxation in complex systems, J. Phys. A: Math. Gen. 36 (2003), 4239-4246.
- [6] Blair A. D., Adelic path integrals, Rev. Math. Physics 7(1995), 21-49.
on a variety, Invent. Math. 77 (1984), 1-23.
- [7] Dynkin E. B., Foundations of the Theory of Markov Processes, Prentice-Hall, Englewood Cliffs, NJ, 1961.

- [8] Evans S. N., Local properties of Lévy processes on a totally disconnected group, *J. Theoret. Probab.* 6 (1993), 817-850.
- [9] Evans S. N., Brownian motion, *J. Theoret. Probab.* 2 (1989), 209-259.
- [10] Friedman A. , *Partial Differential Equations of the Parabolic Type*, Prentice-Hall, New Jersey, 1964.
- [11] Haran S., Potentials and explicit sums in arithmetic, *Invent. Math.* 101(1990), 797-703.
- [12] Haran S., Analytic potential theory over the p -adics, *Ann. Inst. Fourier* 43(1993), 905-944.
- [13] Ilyin A. M. , Kalashnikov A. S., and Oleynik O. A., Linear Second-Order Partial Differential Equations of the Parabolic Type, *J. Math. Sci.* **108** (2002), 435–542.
- [14] R. S. Ismagilov, On the spectrum of the self-adjoint operator in $L_2(p)$ where p is a local field; an analog of the Feynman-Kac formula, *Theor. Math. Phys.* 89(1991), 1024-1028.
- [15] Iordache O., Isopescu R., Isopescu A. , and Frangopol P. T., A non-archimedean theory of interaction of chemical compounds with membranes, *Biosystems*, 24 (1990), 91-98.
- [16] Khrennikov A., p -adic valued distributions in mathematical physics, Kluwer, Dordrecht, 1994.
- [17] Khrennikov A., Non-archimedean analysis: Quantum paradoxes, dynamical systems and biological models, Kluwer, Dordrecht, 1997.
- [18] Kochubei A. N., Pseudodifferential equations and stochastics over non-archimedean fields, Marcel Dekker, 2001.
- [19] Kochubei A. N., Parabolic equations over the field of p -adic numbers, *Math. USSR Izvestiya* 39(1992), 1263-1280.
- [20] Kochubei A. N., Parabolic Pseudodifferential Equations, Hypersingular Integrals, and Markov Processes, *Math. USSR Izvestiya* **33** (1989), 233–259.
- [21] Ladyzhenskaya O. A. , Solonnikov V. A., and Uraltseva N. N., Linear and Quasilinear Equations of Parabolic Type, American Mathematical Society, Providence, 1968.
- [22] Rammal R., and Toulouse G., Ultrametricity for physicists, *Rev. Modern Physics* 58 (1986), 765-778.
- [23] Saia M.J. and Zuniga-Galindo W.A., Local zeta functions for curves, non-degeneracy conditions and Newton polygons, *Trans. Amer. Math. Soc.*, 357 (2005), 59-88.
- [24] Satoh Takakazu, Wiener measures on certain Banach spaces over non-Archimedean local fields. *Compositio Math.* 93 (1994), no. 1, 81–108.
- [25] Taibleson M.H. , Fourier analysis on local fields. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1975.
- [26] Varadarajan V. S., Path integrals for a class of p -adic Schrödinger equations, *Lett. Math. Phys.* 39(1997), 97-106.
- [27] Vladimirov V. S. , On the spectrum of some pseudo-differential operators over p -adic number field, *Algebra and Analysis*, 2, 107-124 (1990).
- [28] Vladimirov V. S., Volovich I. V., and Zelenov E. I. , p -adic Analysis and mathematical physics. Series on Soviet and East European Mathematics, 1. World Scientific Publishing Co., Inc., River Edge, NJ, 1994.
- [29] Zúñiga-Galindo W.A., Parabolic Equations and Markov Processes Over p -adic Fields, accepted in *Potential Analysis*.

DEPARTAMENTO DE MATEMÁTICAS, UNIVERSIDAD NACIONAL DE COLOMBIA, CIUDAD UNIVERSITARIA, BOGOTÁ D.C., COLOMBIA.

E-mail address: jjrodriguezv@unal.edu.co

CENTRO DE INVESTIGACIÓN Y DE ESTUDIOS AVANZADOS DEL I.P.N., DEPARTAMENTO DE MATEMÁTICAS, AV. INSTITUTO POLITÉCNICO NACIONAL 2508, COL. SAN PEDRO ZACATENCO, MÉXICO D.F., C.P. 07360, MÉXICO.

E-mail address: wzuniga@math.cinvestav.mx